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Decay estimates for quasi-linear evolution equations

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Abstract

We consider global strong solutions of the quasi-linear evolution equations (1.1) and (1.2) below, corresponding to sufficiently small initial data, and prove some stability estimates, as $t \to +\infty$, that generalize the corresponding estimates in the linear case.

Résumé

On étudie, quand $t \to +\infty$, le comportement des solutions globales des équations d'évolutions quasi-linéaires (1.1) et (1.2) ci-dessous, pour des données initiales suffisamment petites, généralisant ainsi les estimations correspondantes du cas linéaire.

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1. Introduction

1. In this paper we prove some decay estimates for global, strong solutions of the quasi-linear hyperbolic dissipative equation

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$$u_{tt} + u_t - a_{ij}(Du)\partial_i \partial_j u = 0, \tag{1.1}$$

and its parabolic counterpart

$$u_t - a_{ij}(\nabla u)\partial_i \partial_j u = 0. \tag{1.2}$$

In these equations, $u = u(t, x)$, $t > 0$, $x \in \mathbb{R}^N$, $N \geqslant 3$, and summation for i, j from 1 to N is understood; $u_t := \frac{\partial u}{\partial t}$, $u_{tt} := \frac{\partial^2 u}{\partial t^2}$, $\partial_i := \frac{\partial}{\partial x_i}$, and $Du := \{u_t, \nabla u\}$. We attach to (1.1) and (1.2) the initial conditions (or Cauchy data)

$$u(0, x) = u_0(x), \qquad u_t(0, x) = u_1(x), \tag{1.3}$$

$$u(0, x) = u_0(x), \tag{1.4}$$

where u_0 and u_1 are given functions on $\mathbb{R}^N$. To define strong solutions of (1.1) and (1.2), for $m \in \mathbb{N}$ we denote by $H^m := H^m(\mathbb{R}^N)$ the usual Sobolev spaces of L^2 functions whose distributional derivatives of order up to m are also in L^2. For fixed integer $s > \frac{N}{2} + 1$, we define the spaces

$$\mathcal{Z}_s := C_{\mathrm{b}}^0\big([0, +\infty[; H^{s+1}\big) \cap C_{\mathrm{b}}^1\big([0, +\infty[; H^s\big) \cap C_{\mathrm{b}}^2\big([0, +\infty[; H^{s-1}\big), \tag{1.5}$$

$$\mathcal{P}_s := \big\{u \in C_{\mathrm{b}}\big([0, +\infty[; H^{s+1}\big) \bigm| u_t \in L^2\big(0, +\infty; H^s\big)\big\}. \tag{1.6}$$

Strong solutions of (1.1) and (1.2) are then required to be, respectively, in $\mathcal{Z}_s$ and $\mathcal{P}_s$. Note that, since $H^{s-1} \hookrightarrow C^{0,\alpha}(\mathbb{R}^N)$, for some $\alpha \in]0, 1[$, strong solutions of (1.1) are also classical (i.e., twice differentiable in (t, x)).

2. In Section 2 below, we recall some sufficient conditions on the coefficients a_{ij} and the initial values u_0 and u_1, that guarantee that the Cauchy problems (1.1) + (1.3) and (1.2) + (1.4) do admit unique solutions $u \in \mathcal{Z}_s$ and $u \in \mathcal{P}_s$, which decay to 0, in some specific sense, as $t \to +\infty$ (essentially, u_0 and u_1 have to be "sufficiently small"). In the linear, constant coefficients case, i.e. for the equations

$$u_{tt} + u_t - b_{ij}\partial_i \partial_j u = 0, \tag{1.7}$$

$$u_t - b_{ij}\partial_i \partial_j u = 0, \tag{1.8}$$

with $b_{ij} \in \mathbb{R}$ and $b_{ij}\xi^i \xi^j \geqslant \beta|\xi|^2$ for some $\beta > 0$ and all $\xi \in \mathbb{R}^N$, explicit decay rates of $u(t)$ are known; our goal here is to determine similar rates for the decay of $u(t)$ for the quasi-linear equations (1.1) and (1.2). More precisely, we investigate whether the solutions of (1.1) (resp., (1.2)) may decay with the same rate as the solutions of (1.7) (resp., (1.8)), or whether a loss of decay can be expected. In the hyperbolic case, we find that the decay rates are indeed the same, at least if s is "not too large"; namely, if $\frac{N}{2} + 1 < s \leqslant N$; in the parabolic case, the same holds, with no upper restriction on s. In addition, the decay rates of both types of solutions, in some specific norms, turn out to be the same; as a consequence, one can then study the so-called "diffusion phenomenon" of hyperbolic waves, which consists in showing that, under suitable assumptions on the initial data of (1.1) and (1.2), the difference of the corresponding solutions decays, in the same norm, with a faster rate.

3. Earlier results on the decay rates of solutions of equations which are small perturbations of the linear equation (1.7), that is, of the form

$$u_{tt} + u_t - \Delta u = F\big(\nabla u, \partial_x^2 u\big), \tag{1.9}$$

where F is sufficiently smooth and at least quadratic in a neighborhood of the origin, can be found e.g. in Li Ya-Chun [1]; a first result on the diffusion phenomenon for quasi-linear equations in high space dimension and in the divergence form

$$u_{tt} + u_t - \operatorname{div}\bigl(a(\nabla u)\nabla u\bigr) = 0, \tag{1.10}$$

with a a smooth function satisfying $a(y) = 1 + O(|y|^\alpha)$ as $|y| \to 0$, for some $\alpha \in \mathbb{N}_{>0}$ was given in [9].

2. Recall of previous results

In this section, we recall the global existence results for the quasi-linear Cauchy problems (1.1) + (1.3) and (1.2) + (1.4), which allow us to investigate the long-time behavior of their solutions, and some well-known linear decay estimates for Eqs. (1.7) and (1.8). Given $m \in \mathbb{N}$ and $p \in [1, +\infty]$, we denote by $\|\cdot\|_m$ and $|\cdot|_p$, respectively, the norms in H^m and L^p; we abbreviate $\|\cdot\| = \|\cdot\|_0 = |\cdot|_2$, and denote by $\langle\cdot,\cdot\rangle$ the scalar product in L^2.

1. We assume that the coefficients a_{ij} in (1.1) (resp., (1.2)) are smooth, symmetric, and uniformly strongly elliptic; that is, that $a_{ij} \in C^s(\mathbb{R}^{1+N})$ (resp., $a_{ij} \in C^s(\mathbb{R}^N)$), with $s \in \mathbb{N}$, $s > \frac{N}{2} + 1$, $N \geqslant 3$; $a_{ij}(p) = a_{ji}(p)$ for all $p \in \mathbb{R}^{1+N}$ (resp., for all $p \in \mathbb{R}^N$), and there exists $\alpha_0 > 0$ such that, for all $p \in \mathbb{R}^{1+N}$ and $\xi \in \mathbb{R}^N$,

$$a_{ij}(p)\xi^i\xi^j \geqslant \alpha_0|\xi|^2 \tag{2.1}$$

(resp., for all $p \in \mathbb{R}^N$). Without loss of generality, we assume that $\alpha_0 \geqslant 1$. Under these conditions, we have

Theorem 2.1. *Let $u_0 \in H^{s+1}$, $u_1 \in H^s$. There exists $\delta_0 > 0$ such that, if*

$$\|u_0\|_{s+1} + \|u_1\|_s \leqslant \delta_0, \tag{2.2}$$

the Cauchy problem (1.1) + (1.3) *has a unique solution $u \in \mathcal{Z}_s$, which decays to* 0*, in the sense that*

$$\lim_{t\to+\infty} \underbrace{\bigl(\bigl\|Du(t)\bigr\|_s + \bigl\|u_{tt}(t)\bigr\|_{s-1}\bigr)}_{=:N_h(t)} = 0. \tag{2.3}$$

The quantity $R := \sup_{t\geqslant 0} N_h(t)$ depends on δ_0, and can be made as small as desired by taking δ_0 conveniently small. In turn, the size of δ_0 depends on the ellipticity constant α_0 of (2.1)*, and also on the coefficient of u_t, which in* (1.1) *is* 1.

Theorem 2.2. *Let $u_0 \in H^{s+1}$. There exists $\delta_0 > 0$ such that, if*

$$\|u_0\|_{s+1} \leqslant \delta_0, \tag{2.4}$$

the Cauchy problem (1.2) + (1.4) *has a unique solution $u \in \mathcal{P}_s$, with $u_t \in C_b([0, +\infty[;\ H^{s-1})$, which decays to* 0*, in the sense that*

$$\lim_{t\to+\infty} \underbrace{\bigl(\bigl\|\nabla u(t)\bigr\|_s + \bigl\|u_t(t)\bigr\|_{s-1}\bigr)}_{=:N_p(t)} = 0. \tag{2.5}$$

The quantity $R := \sup_{t\geqslant 0} N_p(t)$ *depends on* δ_0*, and can be made as small as desired by taking* δ_0 *conveniently small. In turn, the size of* δ_0 *depends on the ellipticity constant* α_0 *of* (2.1)*, and also on the coefficient of* u_t*, which in* (1.2) *is* 1.

Theorem 2.1 was essentially proven by Matsumura [3], and later extended (see e.g. Racke [5]) to more general nonlinear, dissipative wave equations of the form

$$u_{tt} + u_t - \partial_j\big(a_{jk}(t,x)\partial_k u\big) = f\big(x,t,u,Du,\nabla u_t,\partial_x^2 u\big). \tag{2.6}$$

The proof of Theorem 2.2 is similar to that of Theorem 2.1. We mention in passing that analogous global existence and decay results can be given for the non-homogeneous versions of (1.1) and (1.2), i.e. for the equation

$$u_{tt} + u_t - a_{ij}(Du)\partial_i\partial_j u = f, \tag{2.7}$$

and the analogous parabolic one, under suitable decay assumptions on f as $t \to +\infty$ (see e.g. [4], for (2.7)).

2. We recall that the quadratic form associated to the coefficients of (1.7) and (1.8) is strongly positive; that is, that (2.1) holds, with $a_{ij}(p)$ replaced by b_{ij}.

2.1. We first consider Eq. (1.7). Given a smooth function $g : \mathbb{R}^N \to \mathbb{R}$, we consider the Cauchy problem consisting of the linear equation (1.7), together with the initial values $u(0) = 0$ and $u_t(0) = g$; we call its solution $u_g(t)$. Given $k, m \in \mathbb{N}$ and $q \in [1,2]$, we set

$$\nu_q(k,m) := \frac{N}{4}\left(\frac{2}{q} - 1\right) + k + \frac{1}{2}m. \tag{2.8}$$

Theorem 2.3. *Let* $r := \max\{k+m-1, 0\}$*, and* $g \in H^r \cap L^q$*. Then, for any multiindex* $\alpha \in \mathbb{N}^N$*, with* $|\alpha| = m$*,*

$$\big\|\partial_t^k\partial_x^\alpha u_g(t)\big\| \leqslant C(1+t)^{-\nu_q(k,m)}\big(\big\|\partial_x^r g\big\| + |g|_q\big). \tag{2.9}$$

In (2.9)*, the constant* C *depends on* N*,* k*,* m*, and* q*.*

Theorem 2.3 is essentially proven in Matsumura [2], for the equation

$$u_{tt} + u_t - \Delta u = 0. \tag{2.10}$$

In this case, u_g can be explicitly found by elementary Fourier transform techniques; Matusmura's estimates also hold for (1.7), since this equation can be transformed into (2.10), by the change of variables $u(t,x) = w(t, B^{1/2}x)$, $B = [b_{ij}]$. In Volkmer [8], it is shown that the rates (2.9) are optimal.

Remark. The decay rates (2.9) show that each derivative of u with respect to t yields the same contribution as any combination of two derivatives with respect to x, as follows from the identity $\nu_q(k, m+2) = \nu_q(k+1, m)$. This is a typically parabolic feature, exemplified by the linear heat equation $u_t = \Delta u$.

It is immediate to show that the solution to the linear Cauchy problem (1.7) + (1.3) is given by $u = u_{u_0+u_1} + \partial_t u_{u_0}$. Then, Theorem 2.3 yields

Theorem 2.4. *Let u be a smooth solution of the Cauchy problem* (1.7) + (1.3); *let k, $m \in \mathbb{N}$, $\alpha \in \mathbb{N}^N$, with $|\alpha| = m$, and $q \in [1, 2]$. Then, u satisfies the estimate*

$$\left\| \partial_t^k \partial_x^\alpha u(t) \right\| \leqslant C(1+t)^{-\nu_q(k,m)} C_{k,m,q}(u, 0), \tag{2.11}$$

where, with $r := \max\{k + m - 1, 0\}$ as in Theorem 2.3,

$$C_{k,m,q}(u, 0) := \left\| \partial_x^{k+m} u(0) \right\| + \left| u(0) \right|_q + \left\| \partial_x^r u_t(0) \right\| + \left| u_t(0) \right|_q. \tag{2.12}$$

In addition, as $t \to +\infty$, $\|u(t)\| \to 0$ as well.

2.2. We now recall the analogous results on the linear parabolic equation (1.8). Again, given a smooth function $g : \mathbb{R}^N \to \mathbb{R}$, we consider the Cauchy problem consisting of the linear equation (1.8), together with the initial value $u(0) = g$; we call its solution $u_g(t)$. Using the properties of the so-called "heat kernel" (see e.g. Racke [6, Chapter 11]), we can prove

Theorem 2.5. *Let $k, m \in \mathbb{N}$, $q \in [1, 2]$ and $\nu_q(k,m)$ be as in* (2.8). *Let $g \in L^q$. Then, for any multiindex α, with $|\alpha| = m$,*

$$\left\| \partial_t^k \partial_x^\alpha u_g(t) \right\| \leqslant C t^{-\nu_q(k,m)} |g|_q, \quad t > 0, \tag{2.13}$$

where C depends on q, k, and m. If in addition $g \in H^{2k+m}$, then for all $t \geqslant 0$,

$$\left\| \partial_t^k \partial_x^\alpha u_g(t) \right\| \leqslant C_1 (1+t)^{-\nu_q(k,m)} \left(\left\| \partial_x^{2k+m} g \right\| + |g|_q \right), \tag{2.14}$$

with $C_1 = 2^{\nu_q(k,m)} C$.

3. The proof of most of the results presented here depends heavily on various technical results, among which we recall the so-called Gagliardo–Nirenberg inequalities and the extension of the chain rule of differentiation to composite functions in Sobolev spaces. A proof of both these results can be found, e.g., in Racke [6].

Proposition 2.1. *Let $m \in \mathbb{N}$, $p, r \in [1, +\infty]$, and $u \in L^p \cap L^r$. Assume that $\partial_x^m u \in L^p$. For integer j, $0 \leqslant j \leqslant m$, and $\theta \in [\frac{j}{m}, 1]$ (with the exception $\theta \neq 1$ if $m - j - \frac{N}{2} \in \mathbb{N}$), define q by*

$$\frac{1}{q} = \frac{j}{N} + \theta \left(\frac{1}{p} - \frac{m}{N} \right) + (1-\theta) \frac{1}{r}. \tag{2.15}$$

Then, for any multi-index $\gamma \in \mathbb{N}^N$, with $|\gamma| = j$, $\partial_x^\gamma u \in L^q$, and satisfies the Gagliardo–Nirenberg inequality

$$\left| \partial_x^\gamma u \right|_q \leqslant C \left| \partial_x^m u \right|_p^\theta |u|_r^{1-\theta}. \tag{2.16}$$

The constant C depends on N, m, j, r, p and θ, but is independent of u.

Given $\varphi \in C^m(\mathbb{R}^M)$, with $m \geqslant 0$ and $M \geqslant 1$, we set

$$h_{m,\varphi}(\lambda) := \max_{|\alpha| \leqslant m} \sup_{|z| \leqslant \lambda} \left| \partial^\alpha \varphi(z) \right|, \quad \lambda \in \mathbb{R}_{\geqslant 0}. \tag{2.17}$$

Proposition 2.2. *Let $m \in \mathbb{N}_{\geqslant 1}$, $p \in [1, +\infty]$, $u = (u^1, \dots, u^M) \in (W^{m,p} \cap C_{\mathrm{b}}(\mathbb{R}^N))^M$, and $\varphi \in C^m(\mathbb{R}^M)$, with $\varphi(0) = 0$. Then, $\varphi(u) \in W^{m,p} \cap C_{\mathrm{b}}(\mathbb{R}^N)$, with*

$$\left|\varphi(u)\right|_p \leqslant C_0 h_{1,\varphi}\big(|u|_\infty\big)|u|_p, \qquad \left|\varphi(u)\right|_\infty \leqslant h_{0,\varphi}\big(|u|_\infty\big). \tag{2.18}$$

In addition, if $p \in]1, +\infty]$, then for $\alpha \in \mathbb{N}^N$, with $1 \leqslant |\alpha| \leqslant m$,

$$\left|\partial^\alpha\big(\varphi(u)\big)\right|_p \leqslant C_0 h_{|\alpha|,\varphi}\big(|u|_\infty\big)\big(1 + |u|_\infty^{m-1}\big)\left|\partial^{|\alpha|}u\right|_p. \tag{2.19}$$

The constant $C_0 \geqslant 1$ depends only on m, N, M, and p.

We will also need the following results.

Lemma 2.1. *Let $\alpha \in \mathbb{R}_{>1}$, $\beta \in \mathbb{R}_{>0}$, and set $\gamma := \min\{\alpha, \beta\}$. There exists $C > 0$, depending on α and β, such that, for all $t \geqslant 0$,*

$$\int_0^t (1 + t - \theta)^{-\alpha}(1 + \theta)^{-\beta}\,\mathrm{d}\theta \leqslant C(1 + t)^{-\gamma}. \tag{2.20}$$

Lemma 2.2. *Let α and $r \in \mathbb{R}_{>0}$. There is $C > 0$, depending on α and r, such that for all $t \geqslant 0$,*

$$\int_0^t (1 + \theta)^{-r}\mathrm{e}^{\alpha\theta}\,\mathrm{d}\theta \leqslant C(1 + t)^{-r}\mathrm{e}^{\alpha t}. \tag{2.21}$$

Proof. Lemma 2.1 is proven in Segal [7]. To prove (2.21), we let $f(t) := (1 + t)^{-r}\mathrm{e}^{\alpha t}$ and $F(t) := \int_0^t f(\theta)\,\mathrm{d}\theta$. Then both $f(t)$ and $F(t) \to +\infty$ as $t \to +\infty$. We compute that

$$\lim_{t\to+\infty} \frac{F'(t)}{f'(t)} = \lim_{t\to+\infty}\left(\alpha - \frac{r}{1+t}\right)^{-1} = \frac{1}{\alpha}; \tag{2.22}$$

hence, by l'Hôpital's rule, $\frac{F(t)}{f(t)} \to \frac{1}{\alpha}$ as $t \to +\infty$. Thus, the function $t \mapsto \frac{F(t)}{f(t)}$, which is continuous and positive on $[0, +\infty[$, admits a maximum value $C = C(\alpha, r)$ on $[0, +\infty[$. This implies (2.21). □

3. Stability estimates

1. For $q = 2$, and $k, m \in \mathbb{N}$, with $0 \leqslant k \leqslant 2$ and $1 \leqslant k + m \leqslant s + 1$, the hyperbolic linear estimate (2.11) reads

$$\left\|\partial_t^k \partial_x^m u(t)\right\| \leqslant C_{k,m}(1 + t)^{-\nu_2(k,m)} \tag{3.1}$$

for all $t \geqslant 0$, where $C_{k,m}$ depends on $\|u_0\|_{k+m}$ and $\|u_1\|_{k+m-1}$. Analogously, the parabolic linear estimate (2.14) can be specialized into

$$\left\|\partial_t^k \partial_x^m u(t)\right\| \leqslant C_{k,m}(1 + t)^{-\nu_2(k,m)}\big(\left\|\partial_x^{2k+m}u_0\right\| + \|u_0\|\big), \tag{3.2}$$

with $q = 2$, $k = 0, 1$, $0 \leqslant 2k + m \leqslant s + 1$. The purpose of this paper is to provide some sufficient conditions, under which solutions of the quasi-linear equations (1.1) and (1.2) decay with the same rates as described in (3.1) and (3.2) for the linear equations (1.7) and (1.8). We assume that the coefficients a_{ij} satisfy the conditions described in §1 of Section 2, and claim:

Theorem 3.1. *Let $N \geqslant 3$, and $s \in \mathbb{N}$, with $\frac{N}{2} + 1 < s \leqslant N$. Assume that $a'_{ij}(0) \neq 0$ and that $u_0 \in H^{s+1} \cap L^1$, $u_1 \in H^s \cap L^1$. There exists $\delta \in]0, 1]$ such that, if*

$$\|u_0\|_{s+1} + |u_0|_1 + \|u_1\|_s + |u_1|_1 \leqslant \delta, \tag{3.3}$$

the corresponding solution of the Cauchy problem (1.1) + (1.3) *is in $\mathcal{Z}_s$, and satisfies the decay estimates*

$$\left\|\partial_x^m \nabla u(t)\right\| \leqslant C_{0,m}(1+t)^{-\nu_2(0,m+1)}, \quad 0 \leqslant m \leqslant s, \tag{3.4}$$

$$\left\|\partial_x^m u_t(t)\right\| \leqslant C_{1,m}(1+t)^{-\nu_2(1,m)}, \quad 0 \leqslant m \leqslant s, \tag{3.5}$$

$$\left\|\partial_x^{m-1} u_{tt}(t)\right\| \leqslant C_{2,m}(1+t)^{-\nu_2(2,m-1)}, \quad 1 \leqslant m \leqslant s. \tag{3.6}$$

In addition,

$$\left\|u(t)\right\| \leqslant C_{0,0}(1+t)^{-\nu_1(0,0)}. \tag{3.7}$$

The constants $C_{k,m}$, $k = 0, 1, 2$, depend on k, m, N, and δ of (3.3).

Theorem 3.2. *Let $N \geqslant 3$, and $s \in \mathbb{N}$, with $s > \frac{N}{2} + 1$. Assume that $a'_{ij}(0) \neq 0$, and that $u_0 \in H^{s+1}$. There exists $\delta \in]0, 1]$ such that, if $\|u_0\|_{s+1} \leqslant \delta$, the corresponding solution of the Cauchy problem* (1.2) + (1.4) *is in $\mathcal{P}_s$, and satisfies the decay estimates*

$$\left\|\partial_x^m \nabla u(t)\right\| \leqslant C_{0,m}(1+t)^{-\nu_2(0,m+1)} \quad \text{for } 0 \leqslant m \leqslant s-1, \tag{3.8}$$

$$\left\|\partial_x^m u_t(t)\right\| \leqslant C_{1,m}(1+t)^{-\nu_2(1,m)} \quad \text{for } 0 \leqslant m \leqslant s-2. \tag{3.9}$$

If $u_0 \in L^1$, with sufficiently small norm, (3.8) *also holds for $m = s$,* (3.9) *holds for $m = s - 1$, and*

$$\left\|u(t)\right\| \leqslant C_{0,0}(1+t)^{-\nu_1(0,0)}. \tag{3.10}$$

The constants $C_{k,m}$, $k = 0, 1$, depend on k, m, N, and δ.

Remarks. Obviously, the conclusions of Theorems 3.1 and 3.2 amplify and complete the information given in (2.3) and (2.5). The decay rates in (3.4), (3.5) and (3.6) (resp., (3.8) and (3.9)) are indeed the same as those in (3.1) (resp., (3.2)) for the linear case. The nonlinear decay rates depend in an essential way on the way in which the functions $p \mapsto \tilde{a}_{ij}(p) := a_{ij}(p) - a_{ij}(0)$, $p \in \mathbb{R}^{1+N}$, vanish as $p \to 0$, as measured (e.g.) by an estimate of the type

$$\left|a_{ij}(p) - a_{ij}(0)\right| \leqslant C|p|^\rho, \quad |p| \leqslant \delta, \tag{3.11}$$

for some $\rho \geqslant 1$ and C, $\delta > 0$. Note that (3.11) is trivially satisfied at least for $\rho = 1$, as follows from the mean value theorem; in fact, the assumption $a'_{ij}(0) \neq 0$ in Theorems 3.1 and 3.2 means that (3.11) is satisfied only for $\rho = 1$. The upper limitation $s \leqslant N$ in Theorem 3.1 is technical; while it seems unnatural, we ignore if it is actually necessary. However, the possible relaxing of such limitation seems to depend, at least in part, on conditions like (3.11), with $\rho > 1$. Finally, we note that, comparing (3.6) for $m = s$ with (3.4) for $m = s$ and (3.5) for $m = s - 1$, we see that, in L^2, $\partial_x^{s-1} u_t(t)$ and $\partial_x^{s-1}\partial_i\partial_j u(t)$ decay with the same rate $r = \frac{s+1}{2}$, which is the same decay rate of $\partial_x^{s-1} u_t(t)$ and $\partial_x^{s-1}\partial_i\partial_j u(t)$ in the parabolic case, while $\partial_x^{s-1} u_{tt}(t)$ decays at the *faster* rate $r + 1 = \frac{s+3}{2}$. This motivates the diffusion phenomenon conjecture that the asymptotic profile of the solutions of (1.1) should "coincide" with that of a solution of the parabolic equation (1.2).

4. Proof of Theorem 3.1

If δ of (3.3) is sufficiently small, Theorem 2.1 implies that the Cauchy problem (1.1) + (1.3) does have a global solution $u \in \mathcal{Z}_s$. In addition, if $R := \sup_{t\geqslant 0} N_h(t)$, we know that for all $R_1 > 0$ there exists $\delta_0 \in]0,1[$ such that $R \leqslant R_1$ if $\delta \leqslant \delta_0$; in particular, we can assume that $R \leqslant 1$. We also note that it is sufficient to establish the decay estimates for the lowest and highest indicated values of m, the intermediate values following by interpolation, via the Gagliardo–Nirenberg inequalities. In the sequel, we omit to record the dependence of the constants $C_{k,m}$ on the indices k, m, etc.

We proceed in five steps. In Step 1, we prove (3.4) and (3.5) for $m = 0$, and (3.7). In Step 2, we prove an intermediate estimate on $|\partial_x^{s-k}\partial_t^k u(t)|_2$, $0 \leqslant k \leqslant 3$ (recall that $s \geqslant 3$ because $N \geqslant 3$), in terms of $|\partial_x^s Du(t)|_2$. In Step 3, we use this estimate, as part of an "energy estimates" procedure, to prove (3.4) for $m = s$; as a consequence, we deduce (3.5) and (3.6) for $m \leqslant s-1$. In Step 4, we use a similar "energy" method, to prove (3.5) and (3.6) for $m = s$. In Step 5 we prove two estimates, which are used in Steps 3 and 4. In Steps 1 and 2, we apply the linear estimates of Theorem 2.3 to the representation of the solution of (1.1) in integral form, by means of Duhamel's formula. More precisely, we rewrite Eq. (1.1) in the linearized form

$$u_{tt} + u_t - a_{ij}(0)\partial_i\partial_j u = g(u) := \tilde{a}_{ij}(Du)\partial_i\partial_j u, \tag{4.1}$$

set $b_{ij} := a_{ij}(0)$, and decompose the solution of (1.1) as $u(t) = v(t) + w(t)$, where $v(t) = u_{u_0+u_1} + \partial_t u_{u_0}$ and

$$w(t) := \int_0^t u_{g(u(\theta))}(t-\theta)\,\mathrm{d}\theta. \tag{4.2}$$

Then, by the very definition of u_g,

$$w_t(t) = \int_0^t \partial_t\big(u_{g(u(\theta))}(t-\theta)\big)\,\mathrm{d}\theta; \tag{4.3}$$

in addition, as we shall need in Steps 3 and 4,

$$w_{tt}(t) = g\big(u(t)\big) + \int_0^t \partial_{tt}\big(u_{g(u(\theta))}(t-\theta)\big)\,\mathrm{d}\theta, \tag{4.4}$$

$$w_{ttt}(t) = \partial_t g\big(u(t)\big) - g\big(u(t)\big) + \int_0^t \partial_t^3\big(u_{g(u(\theta))}(t-\theta)\big)\,\mathrm{d}\theta. \tag{4.5}$$

The decomposition $u = v + w$ shows that we can expect u to decay, at best, at a rate as fast as that of v (that is, the linear estimates (3.1)), and that, in fact, it is sufficient to show the decay estimates (3.4), (3.5) and (3.6) for w.

In the sequel, we denote by $\mathcal{K}_0$ the set of all continuous, non-decreasing functions $\omega : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$, such that $\omega(0) = 0$.

Step 1. We prove (3.4) for $m = 0$, by showing that u satisfies the faster decay estimate

$$\|Du(t)\| \leqslant C(1+t)^{-\nu_1(0,1)} = C(1+t)^{-(N/4+1/2)}. \tag{4.6}$$

To this end, we set

$$\Phi_0(t) := \sup_{0\leqslant\theta\leqslant t} \left((1+\theta)^{\nu_1(0,1)} \|Du(\theta)\|\right), \tag{4.7}$$

and prove that Φ_0 is bounded. Since $\nu_1(0,1) \leqslant \nu_1(1,0)$, we obtain from (4.2) and (4.3) that

$$\left|Dw(t)\right|_2 \leqslant C \int_0^t (1+t-\theta)^{-\nu_1(0,1)} \left(\left|g(u)\right|_2 + \left|g(u)\right|_1\right) \mathrm{d}\theta. \tag{4.8}$$

By Proposition 2.2 (note that $\tilde{a}_{ij}(0) = 0$), we obtain that

$$\begin{aligned}
\left|g(u)\right|_2 + \left|g(u)\right|_1 &\leqslant \left|\tilde{a}_{ij}(Du)\right|_2 \left(|\partial_i\partial_j u|_\infty + |\partial_i\partial_j u|_2\right) \\
&\leqslant \kappa(R)|Du|_2 \left(\left|\partial_x^2 u\right|_\infty + \left|\partial_x^2 u\right|_2\right) \\
&\leqslant \omega(R)|Du|_2 \\
&\leqslant \omega(R)\Phi_0(\theta)(1+\theta)^{-\nu_1(0,1)},
\end{aligned} \tag{4.9}$$

with $\omega \in \mathcal{K}_0$. We replace (4.9) into (4.8); since Φ_0 is increasing, and $\nu_1(0,1) > 1$ because $N \geqslant 3$, Lemma 2.1 implies that

$$\begin{aligned}
\left|Dw(t)\right|_2 &\leqslant \omega(R)\Phi_0(t) \int_0^t (1+t-\theta)^{-\nu_1(0,1)}(1+\theta)^{-\nu_1(0,1)} \mathrm{d}\theta \\
&\leqslant \omega(R)\Phi_0(t)(1+t)^{-\nu_1(0,1)}.
\end{aligned} \tag{4.10}$$

Since u_0 and $u_1 \in L^1$, v satisfies the linear estimate

$$\left|Dv(t)\right|_2 \leqslant C_L(1+t)^{-\nu_1(0,1)}; \tag{4.11}$$

hence, we conclude from (4.11) and (4.10) that

$$(1+t)^{\nu_1(0,1)}\left|Du(t)\right|_2 \leqslant C_L + \omega(R)\Phi_0(t). \tag{4.12}$$

Choosing R so small that $\omega(R) \leqslant \frac{1}{2}$ (which can be done by choosing δ sufficiently small), and recalling that Φ_0 is non-decreasing, we deduce from (4.12) that

$$\Phi_0(t) \leqslant 2C_L. \tag{4.13}$$

Thus, Φ_0 is bounded, and (3.4) follows for $m = 0$. Note also that, replacing (4.13) into (4.12) yields (4.6). We can then proceed to prove (3.5) for $m = 0$, and (3.7). Indeed, from (4.3), (4.9) and (4.13) we obtain that

$$\begin{aligned}
\left|w_t(t)\right|_2 &\leqslant C \int_0^t (1+t-\theta)^{-\nu_1(1,0)} \left(\left|g(u)\right|_2 + \left|g(u)\right|_1\right) \mathrm{d}\theta \\
&\leqslant C\omega(R) \int_0^t (1+t-\theta)^{-\nu_1(1,0)} \Phi_0(\theta)(1+\theta)^{-\nu_1(0,1)} \mathrm{d}\theta \\
&\leqslant C(1+t)^{-\nu_1(0,1)} \leqslant C(1+t)^{-\nu_2(1,0)}.
\end{aligned} \tag{4.14}$$

Since $|v_t(t)|_2$ satisfies a similar estimate, (3.5) for $m=0$ follows. In the same way, we derive from (4.2) and (2.9), with $r=0$, that

$$\left|w(t)\right|_2 \leqslant C\int_0^t (1+t-\theta)^{-\nu_1(0,0)}\left(\left|g(u)\right|_2+\left|g(u)\right|_1\right)\mathrm{d}\theta. \tag{4.15}$$

By (4.9), (4.13) and Lemma 2.1, we deduce then that

$$\left|w(t)\right|_2 \leqslant 2C_L C\omega(R)(1+t)^{-\nu_1(0,0)}, \tag{4.16}$$

from which, since $v(t)$ satisfies a similar estimate, (3.7) follows.

Step 2. 1) We set

$$\Phi_s(t) := \max\Big\{1, \sup_{0\leqslant\theta\leqslant t}\left((1+\theta)^{\nu_2(0,s+1)}\left\|\partial_x^s Du(\theta)\right\|\right)\Big\} \tag{4.17}$$

(note that Φ_s is continuous), and prove the intermediate estimates

$$\left|\partial_x^{s-k}\partial_t^k u(t)\right|_2 \leqslant \left(C_k+\omega_k(R)\left(\Phi_s(t)\right)^2\right)(1+t)^{-\nu_2(k,s-k)}, \tag{4.18}$$

for $0\leqslant k\leqslant 2$, and $\omega_k\in\mathcal{K}_0$, and

$$\left|\partial_x^{s-3}\partial_t^3 u(t)\right|_2 \leqslant C_3\left(\Phi_s(t)\right)^7(1+t)^{-\nu_2(3,s-3)}. \tag{4.19}$$

In (4.18) and (4.19), the constants C_k are independent of R; in fact, they depend on the constants $C_{k,s-k,2}$ of (2.11), and are, therefore, of the type $\omega(\delta)$, with $\omega\in\mathcal{K}_0$ and δ as in (3.3). For convenience, in each of the estimates that follow, we use the same letters λ, μ and ν to denote various exponents, that are different within each estimate; since the context is clear, this should lead to no confusion. Likewise, for $m\in\mathbb{N}$ we write $\Phi_s^m(t)$ instead of $(\Phi_s(t))^m$ and, e.g., $\partial_x^m g(u)$, $\partial_t^m g(u)$, instead of $\partial_x^m(g(u))$ and $\partial_t^m(g(u))$.

From (4.2), (4.3) and (4.4), we see that, for $0\leqslant k\leqslant 3$,

$$\left|\partial_x^{s-k}\partial_t^k w(t)\right|_2 \leqslant \left|\partial_x^{s-k}\Gamma_k(u)\right|_2 + C\int_0^t \underbrace{(1+t-\theta)^{-\nu_2(k,s-k)}\left(\left|\partial_x^{s-1}g(u)\right|_2+\left|g(u)\right|_2\right)}_{=:I_k(t,\theta)}\mathrm{d}\theta, \tag{4.20}$$

with $\Gamma_0(u)=\Gamma_1(u)=0$, $\Gamma_2(u)=g(u)$, and $\Gamma_3(u)=\partial_t g(u)-2g(u)$. Recalling (4.6), and that Φ_s is non-decreasing, by Proposition 2.1 we can modify the estimate of $|g(u)|_2$ in (4.9) into

$$\begin{aligned}\left|g\big(u(\theta)\big)\right|_2 &\leqslant \left|\tilde a_{ij}(Du)\right|_2|\partial_i\partial_j u|_\infty\\ &\leqslant \kappa(R)|Du|_2\left|\partial_x^s\nabla u\right|_2^{\lambda}|\nabla u|_2^{1-\lambda}\\ &\leqslant \omega(R)\Phi_s^{\lambda-\varepsilon}(\theta)(1+\theta)^{-\nu+\varepsilon(s+1)/2},\end{aligned} \tag{4.21}$$

with $\lambda=\frac{N+2}{2s}\in\,]\frac{1}{s},1[$, $\varepsilon\in\,]0,\lambda[$ to be determined, and

$$\nu := \left(\frac{N}{4}+\frac{1}{2}\right)(2-\lambda)+\frac{s+1}{2}\lambda = \frac{3(N+2)}{4}-\frac{N(N+2)}{8s}. \tag{4.22}$$

The assumption $\frac{N}{2}+1<s\leqslant N$ and the limitation $k\leqslant 3$ imply that $\nu>\frac{s+3}{2}\geqslant \nu_2(k,s-k)$. Thus, we can choose $\varepsilon>0$ such that $\nu-\frac{\varepsilon}{2}(s+1)\geqslant \nu_2(k,s-k)$, and deduce from (4.21) that

$$\bigl|g\bigl(u(\theta)\bigr)\bigr|_2\leqslant \omega(R)\Phi_s^{\lambda-\varepsilon}(\theta)(1+\theta)^{-\nu_2(k,s-k)}. \tag{4.23}$$

Since $\Phi_s(t)\geqslant 1$, we conclude that

$$\bigl|g\bigl(u(\theta)\bigr)\bigr|_2\leqslant \omega(R)\Phi_s(\theta)(1+\theta)^{-\nu_2(k,s-k)}. \tag{4.24}$$

2) Going back to (4.20), by Leibniz' formula we decompose

$$\partial_x^{s-1}g(u)=\sum_{|\alpha|=s-1}\sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}\underbrace{\partial_x^\beta\tilde a_{ij}(Du)\partial_x^{\alpha-\beta}\partial_i\partial_j u}_{=:\Lambda_{\alpha\beta}}. \tag{4.25}$$

By means of the Hölder and Gagliardo–Nirenberg inequalities, as in (4.21), we can find numbers $p,q\geqslant 2$, such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$ and

$$|\Lambda_{\alpha\beta}|_2\leqslant \kappa(R)\bigl|\partial_x^{|\beta|}Du\bigr|_p\bigl|\partial_x^{s-|\beta|}\nabla u\bigr|_q\leqslant C\bigl|\partial_x^s Du\bigr|_2^\lambda|Du|_2^{2-\lambda}, \tag{4.26}$$

where $\lambda=1+\frac{N}{2s}\in\,]1,2[$ (this procedure is explained in larger detail in the estimate of (4.39) below, to which we refer). Recalling (3.4) and (3.5) for $m=0$, we obtain from (4.26), and Step 1 of this proof, that, for $\varepsilon=\frac{1}{s+1}$,

$$|\Lambda_{\alpha\beta}|_2\leqslant C\bigl|\partial_x^s Du\bigr|_2^\varepsilon\Phi_s^{\lambda-\varepsilon}(\theta)(1+\theta)^{-\nu}, \tag{4.27}$$

with

$$\nu:=\left(1+\frac{N}{2s}-\frac{1}{s+1}\right)\frac{s+1}{2}+\left(\frac{N}{4}+\frac{1}{2}\right)\left(1-\frac{N}{2s}\right). \tag{4.28}$$

Then, $\lambda-\varepsilon\leqslant 1+\frac{N}{2s}\leqslant 2$, and $\nu\geqslant\frac{s+3}{2}\geqslant\nu_2(k,s-k)$ (because of the assumption $\frac{N}{2}+1<s\leqslant N$); consequently, from (4.25),

$$\bigl|\partial_x^{s-1}g\bigl(u(\theta)\bigr)\bigr|_2\leqslant \omega(R)\Phi_s^2(\theta)(1+\theta)^{-\nu_2(k,s-k)}. \tag{4.29}$$

Together with (4.24), (4.29) implies, via Lemma 2.1, that

$$\int_0^t I_k(t,\theta)\,\mathrm{d}\theta\leqslant \omega(R)\Phi_s^2(t)(1+t)^{-\nu_2(k,s-k)}. \tag{4.30}$$

When $k=2$, we estimate $|\partial_x^{s-2}\Gamma_2(u)|_2=|\partial_x^{s-2}g(u)|_2$ by interpolation from (4.29) and (4.24); recalling that $\Phi_s(t)\geqslant 1$, we obtain

$$\bigl|\partial_x^{s-2}\Gamma_2(u)\bigr|_2\leqslant \omega(R)\Phi_s^2(\theta)(1+\theta)^{-\nu_2(k,s-k)}. \tag{4.31}$$

Inserting (4.30) and (4.31) into (4.20), we conclude that, if $0\leqslant k\leqslant 2$,

$$\bigl|\partial_x^{s-k}\partial_t^k w(t)\bigr|_2\leqslant \omega(R)\Phi_s^2(t)(1+t)^{-\nu_2(k,s-k)}. \tag{4.32}$$

Adding this to the analogous estimate for $|\partial_x^{s-k}\partial_t^k v(t)|_2$, (4.18) follows. When $k=3$, we still need to estimate

$$\partial_x^{s-3}\partial_t\big(g(u)\big) = \partial_x^{s-3}\big(\tilde a_{ij}(Du)\partial_i\partial_j u_t\big) + \partial_x^{s-3}\big(\tilde a'_{ij}(Du)\cdot Du_t\partial_i\partial_j u\big) =: S_1 + S_2. \tag{4.33}$$

By Leibniz' formula,

$$S_1 = \sum_{|\alpha|=s-3}\sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}\partial_x^\beta\tilde a_{ij}(Du)\partial_x^{\alpha-\beta}\partial_i\partial_j u_t; \tag{4.34}$$

acting as in (4.26), we can find numbers $p, q \geqslant 2$, such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$ and

$$|S_1|_2 \leqslant C\sum_{|\beta|\leqslant s-3}\big|\partial_x^{|\beta|}\tilde a_{ij}(Du)\big|_p\big|\partial_x^{s-1-|\beta|}u_t\big|_q \leqslant C\big|\partial_x^s Du\big|_2^\mu|Du|_2^{2-\mu}, \tag{4.35}$$

with $\mu = 1+\frac{N-2}{2s}\in\,]1,2[$. Thus, we obtain from (4.35), using (4.6),

$$|S_1|_2 \leqslant \omega(R)\Phi_s(t)(1+t)^{-\nu} \leqslant C\Phi_s(t)(1+t)^{-\nu_2(3,s-3)}, \tag{4.36}$$

with

$$\nu := \frac{s+1}{2}+\left(1-\frac{N-2}{2s}\right)\left(\frac{N}{4}+\frac{1}{2}\right) \geqslant \frac{s+3}{2} = \nu_2(3,s-3). \tag{4.37}$$

Likewise,

$$S_2 = \sum_{|\alpha|=s-3}\sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}\sum_{\gamma\leqslant\beta}\binom{\beta}{\gamma}\underbrace{\partial_x^\gamma\tilde a'_{ij}(Du)\partial_x^{\beta-\gamma}Du_t\partial_x^{\alpha-\beta}\partial_i\partial_j u}_{=:S_2^{\alpha\beta\gamma}}. \tag{4.38}$$

When $\gamma > 0$, again by Proposition 2.1 we can estimate

$$\big|S_2^{\alpha\beta\gamma}\big|_2 \leqslant \kappa(R)\big|\partial_x^{|\gamma|}Du\big|_p\big|\partial_x^{|\beta|-|\gamma|}Du_t\big|_q\big|\partial_x^{s-2-|\beta|}\nabla u\big|_r, \tag{4.39}$$

with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=\frac{1}{2}$,

$$\big|\partial_x^{|\gamma|}Du\big|_p \leqslant C\big|\partial_x^{s-1}Du\big|_2^\lambda|Du|_2^{1-\lambda}, \tag{4.40}$$

$$\big|\partial_x^{|\beta|-|\gamma|}Du_t\big|_q \leqslant C\big|\partial_x^{s-1}Du_t\big|_2^\mu|Du_t|_2^{1-\mu}, \tag{4.41}$$

$$\big|\partial_x^{s-2-|\beta|}\nabla u\big|_r \leqslant C\big|\partial_x^{s-1}\nabla u\big|_2^\sigma|\nabla u|_2^{1-\sigma}, \tag{4.42}$$

and

$$\frac{1}{p} = \frac{|\gamma|}{N}+\frac{1}{2}-\lambda\frac{s-1}{N}, \qquad \frac{1}{q} = \frac{|\beta|-|\gamma|}{N}+\frac{1}{2}-\mu\frac{s-1}{N},$$
$$\frac{1}{r} = \frac{s-2-|\beta|}{N}+\frac{1}{2}-\sigma\frac{s-1}{N}. \tag{4.43}$$

In (4.40) and (4.42), we use (4.18) for $k=0$ and $k=1$ to estimate

$$\big|\partial_x^{s-1}\nabla u\big|_2 \leqslant \big|\partial_x^{s-1}Du\big|_2 \leqslant \big(C+\omega(R)\Phi_s^2(t)\big)(1+t)^{-\nu_2(0,s)}, \tag{4.44}$$

for suitable $C>0$ and $\omega\in\mathcal{K}_0$. In (4.41), using Eq. (1.1) itself we first estimate

$$\begin{aligned}|Du_t|_2 &\leqslant |\nabla u_t|_2 + |u_{tt}|_2\\
&\leqslant |\nabla u_t|_2 + |u_t|_2 + \left|a_{ij}(Du)\right|_\infty |\partial_i\partial_j u|_2\\
&\leqslant C\left|\partial_x^s u_t\right|_2^{1/s}|u_t|_2^{1-1/s} + |u_t|_2 + \kappa(R)\left|\partial_x^s\nabla u\right|_2^{1/s}|\nabla u|_2^{1-1/s}.\end{aligned} \tag{4.45}$$

Recalling (4.6), we deduce from (4.45) that

$$|Du_t|_2 \leqslant C\Phi_s^{1/s}(t)(1+t)^{-\alpha} + C(1+t)^{-\nu_1(0,1)}, \tag{4.46}$$

with $\alpha = \frac{s+1}{2s} + (1-\frac{1}{s})(\frac{N}{4}+\frac{1}{2})$; and, since $\Phi_s(t) \geqslant 1$ and $\frac{s+1}{2s} \geqslant \frac{1}{s}(\frac{N}{4}+\frac{1}{2})$, we finally obtain that

$$\left|Du_t(t)\right|_2 \leqslant C\Phi_s(t)(1+t)^{-\nu_1(0,1)}. \tag{4.47}$$

Similarly,

$$\begin{aligned}\left|\partial_x^{s-1}Du_t\right|_2 &\leqslant \left|\partial_x^{s-1}\nabla u_t\right|_2 + \left|\partial_x^{s-1}u_{tt}\right|_2\\
&\leqslant \left|\partial_x^s u_t\right|_2 + \left|\partial_x^{s-1}\big(a_{ij}(Du)\partial_i\partial_j u - u_t\big)\right|_2\\
&\leqslant \left|\partial_x^s u_t\right|_2 + \left|\partial_x^{s-1}u_t\right|_2 + \left|\partial_x^{s-1}g(u)\right|_2 + \left|\partial_x^{s-1}\big(a_{ij}(0)\partial_i\partial_j u\big)\right|_2.\end{aligned} \tag{4.48}$$

By (4.18) and (4.29), we can proceed with

$$\begin{aligned}\left|\partial_x^{s-1}Du_t\right|_2 &\leqslant \big(1+\left|a_{ij}(0)\right|\big)\left|\partial_x^s Du\right|_2 + \left|\partial_x^{s-1}u_t\right|_2 + \left|\partial_x^{s-1}g(u)\right|_2\\
&\leqslant C\Phi_s(t)(1+t)^{-\nu_2(0,s+1)} + \big(C_1 + \omega_1(R)\Phi_s^2(t)\big)(1+t)^{-\nu_2(1,s-1)}\\
&\quad + \big(C + \omega(R)\Phi_s(t)\big)(1+t)^{-\nu_2(3,s-3)}\\
&\leqslant C\Phi_s^2(t)(1+t)^{-\nu_2(0,s+1)}.\end{aligned} \tag{4.49}$$

Consequently, replacing (4.44), (4.47) and (4.49) into (4.40), (4.41), (4.42), and then into (4.39), we obtain that

$$\left|S_2^{\alpha\beta\gamma}\right|_2 \leqslant C\Phi_s^{2(\lambda+\mu+\sigma)+1-\mu}(t)(1+t)^{-\nu}, \tag{4.50}$$

where

$$\nu := \frac{s}{2}(\lambda+\mu+\sigma) + \frac{1}{2}\mu + \left(\frac{N}{4}+\frac{1}{2}\right)\big(3-(\lambda+\mu+\sigma)\big). \tag{4.51}$$

We compute that $\lambda+\mu+\sigma = 1 + \frac{N-1}{s-1} \in [2,3[$, again because $\frac{N}{2}+1 < s \leqslant N$; thus, since also $N \geqslant 3$,

$$\begin{aligned}\nu &\geqslant \left(1+\frac{N-1}{s-1}\right)\left(\frac{s}{2}-\frac{N}{4}-\frac{1}{2}\right) + \frac{3N}{4} + \frac{3}{2}\\
&\geqslant \frac{s+3}{2} + \frac{N-1}{2(s-1)}\left(s-\left(\frac{N}{2}+1\right)\right) + \frac{N}{2} - \frac{1}{2}\\
&\geqslant \frac{s+3}{2} = \nu_2(3,s-3).\end{aligned} \tag{4.52}$$

Consequently, we deduce from (4.50) that, if $\gamma > 0$,

$$\left|S_2^{\alpha\beta\gamma}\right|_2 \leqslant C\Phi_s^7(t)(1+t)^{-\nu_2(3,s-3)}. \tag{4.53}$$

If $\gamma = 0$, we replace (4.39) with

$$\left|S_2^{\alpha\beta 0}\right|_2 \leqslant \kappa(R)\left|\partial_x^{|\beta|} Du_t\right|_q \left|\partial_x^{s-2-|\beta|}\nabla u\right|_r, \tag{4.54}$$

with, now, $\frac{1}{q}+\frac{1}{r}=\frac{1}{2}$. Proceeding as in (4.41) and (4.42), and recalling (4.47) and (4.49), we obtain that

$$\begin{aligned}\left|S_2^{\alpha\beta 0}\right|_2 &\leqslant C\left|\partial_x^{s-1} Du_t\right|_2^{\mu} |Du_t|_2^{1-\mu}\left|\partial_x^{s-1}\nabla u\right|_2^{\sigma}|\nabla u|_2^{1-\sigma}\\ &\leqslant C\Phi_s^{2(\mu+\sigma)+1-\mu}(t)(1+t)^{-\nu},\end{aligned} \tag{4.55}$$

where $\mu+\sigma = 1+\frac{N-2}{2(s-1)} \in \left]1,2\right[$, and, now,

$$\begin{aligned}\nu &:= \frac{s}{2}(\mu+\sigma)+\left(\frac{N}{4}+\frac{1}{2}\right)\big(2-(\mu+\sigma)\big)+\frac{1}{2}\mu\\ &\geqslant \left(1+\frac{N-2}{2(s-1)}\right)\left(\frac{s}{2}-\frac{1}{2}-\frac{N}{4}\right)+\frac{N}{2}+1\\ &\geqslant \frac{s}{2}+\frac{1}{2}+\frac{N}{4}+\frac{N-2}{4(s-1)}\left(s-\frac{N}{2}-1\right).\end{aligned} \tag{4.56}$$

If $N \geqslant 4$, we proceed with $\nu \geqslant \frac{s}{2}+\frac{3}{2} = \nu_2(3,s-3)$, and we deduce that

$$\left|S_2^{\alpha\beta 0}\right|_2 \leqslant C\Phi_s^5(t)(1+t)^{-\nu_2(3,s-3)}. \tag{4.57}$$

If $N=3$, the conditions $\frac{3}{2}+1<s\leqslant 3$ imply that $s=3$ as well; thus, recalling (4.38), the sum S_2 reduces to the single term S_2^{000}. Recalling (4.47) and (4.49), as well as (4.6) and the definition (4.17) of Φ_s, and that $\nu_2(3,0)=3$, by Proposition 2.1 we obtain

$$\begin{aligned}\left|S_2^{000}\right|_2 &\leqslant \left|\tilde{a}_{ij}(Du)\right|_\infty |Du_t|_\infty |\partial_i\partial_j u|_2\\ &\leqslant \kappa(R)|Du_t|_\infty\left|\partial_x^2 u\right|_2\\ &\leqslant C\left|\partial_x^2 Du_t\right|_2^{3/4}|Du_t|_2^{1/4}\left|\partial_x^3\nabla u\right|_2^{1/3}|\nabla u|_2^{2/3}\\ &\leqslant C\Phi_3^p(t)(1+t)^{-q},\end{aligned} \tag{4.58}$$

with specific $p\in\left]1,2\right[$ and $q\in\left]3,4\right[$. Thus, we conclude from (4.58) that

$$\left|S_2^{000}\right|_2 \leqslant C\Phi_s^2(t)(1+t)^{-\nu_2(3,s-3)}, \quad s=3. \tag{4.59}$$

Recalling (4.33), we deduce from (4.53), (4.57) and (4.59), together with (4.36), that

$$\left|\partial_x^{s-3}\partial_t g(u)\right|_2 \leqslant C\Phi_s^7(t)(1+t)^{-\nu_2(3,s-3)}. \tag{4.60}$$

Estimating $|\partial_x^{s-3}g(u)|_2$ again by interpolation, as we did to obtain (4.31), we finally conclude that $\Gamma_3(u)$ (defined after (4.20)) satisfies the estimate

$$\left|\partial_x^{s-3}\Gamma_3(u)\right|_2 \leqslant C\Phi_s^7(\theta)(1+\theta)^{-\nu_2(3,s-3)}. \tag{4.61}$$

Inserting (4.61) and (4.30) into (4.20) (for $k=3$), and adding the corresponding linear estimate for $\partial_x^{s-3}\partial_t^3 v$, we finally conclude the proof of (4.19).

Step 3. 1) We show that the intermediate estimate (4.18) for $k=1$, i.e.

$$\left|\partial_x^{s-1}u_t(t)\right|_2 \leqslant \big(C_1+\omega_1(R)\Phi_s^2(t)\big)(1+t)^{-\nu_2(1,s-1)}, \tag{4.62}$$

allows us to prove, by means of an energy estimates argument, that Φ_s is bounded; in turn, this implies (3.4) for $m = s$. We set

$$Q_s(\nabla u) := \sum_{|\alpha| \leqslant s} \langle a_{ij}(Du)\partial_i \partial_x^\alpha u, \partial_j \partial_x^\alpha u \rangle, \tag{4.63}$$

$$P^u(\cdot) := |\partial_x^s u_t|_2^2 + \langle \partial_x^s u, \partial_x^s u_t \rangle + Q_s(\nabla u). \tag{4.64}$$

We differentiate Eq. (1.1) α times with respect to x, $|\alpha| = s$, and multiply the resulting equations in L^2 by $2\partial_x^\alpha u_t + \partial_x^\alpha u$. Summing the resulting identities for $|\alpha| = s$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} P^u + P^u = \sum_{|\alpha|=s} (R_{0,\alpha} + R_{1,\alpha} + R_{2,\alpha} + R_{G,1,\alpha} + R_{G,2,\alpha}) =: \Lambda_0, \tag{4.65}$$

where

$$R_{0,\alpha} := \langle a'_{ij}(Du) \cdot Du_t \partial_i \partial_x^\alpha u, \partial_j \partial_x^\alpha u \rangle, \tag{4.66}$$

$$R_{1,\alpha} := -2\langle a'_{ij}(Du) \cdot D\partial_j u \partial_i \partial_x^\alpha u, \partial_x^\alpha u_t \rangle, \tag{4.67}$$

$$R_{2,\alpha} := -2\langle a'_{ij}(Du) \cdot D\partial_j u \partial_i \partial_x^\alpha u, \partial_x^\alpha u \rangle, \tag{4.68}$$

$$R_{G,1,\alpha} := 2\langle G_\alpha(u), \partial_x^\alpha u_t \rangle, \tag{4.69}$$

$$R_{G,2,\alpha} := 2\langle G_\alpha(u), \partial_x^\alpha u \rangle, \tag{4.70}$$

with

$$G_\alpha(u) := \sum_{0<\beta\leqslant\alpha} \binom{\alpha}{\beta} \partial_x^\beta a_{ij}(Du) \partial_x^{\alpha-\beta} \partial_i \partial_j u. \tag{4.71}$$

In Step 5, we shall prove that there is $\omega \in \mathcal{K}_0$ such that

$$\Lambda_0 \leqslant \omega(R) |\partial_x^s Du|_2^2 + \omega(R) \Phi_s^3(t)(1+t)^{-2\nu_2(0,s+1)}. \tag{4.72}$$

Assuming this for the moment, we deduce from (4.65) that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^t P^u) &\leqslant \mathrm{e}^t \omega(R) \big(|\partial_x^s Du|_2^2 + \Phi_s^3(t)(1+t)^{-2\nu_2(0,s+1)} \big) \\ &\leqslant \omega(R) \mathrm{e}^t \big(\Phi_s^2(t) + \Phi_s^3(t) \big)(1+t)^{-2\nu_2(0,s+1)}. \end{aligned} \tag{4.73}$$

Since $\Phi_s \geqslant 1$, we deduce from (4.73) that

$$\mathrm{e}^t P^u(t) \leqslant P^u(0) + \omega(R) \Phi_s^3(t) \int_0^t \mathrm{e}^\theta (1+\theta)^{-2\nu_2(0,s+1)} \, \mathrm{d}\theta. \tag{4.74}$$

Recalling that, by (3.3), $|P^u(0)| \leqslant C\delta^2 \leqslant C$, by Lemma 2.2 we deduce from (4.74) that

$$P^u(t) \leqslant \big(C + \omega(R)\Phi_s^3(t) \big)(1+t)^{-2\nu_2(0,s+1)} =: Z_0(t). \tag{4.75}$$

Recalling (4.64), we further deduce from (4.75) that

$$\begin{aligned} |\partial_x^s Du(t)|_2^2 &\leqslant Z_0(t) - \langle \partial_x^s u(t), \partial_x^s u_t(t) \rangle \\ &\leqslant Z_0(t) + \langle \partial_x^{s+1} u(t), \partial_x^{s-1} u_t(t) \rangle \\ &\leqslant Z_0(t) + \frac{1}{2} |\partial_x^s \nabla u(t)|_2^2 + \frac{1}{2} |\partial_x^{s-1} u_t(t)|_2^2, \end{aligned} \tag{4.76}$$

from which

$$\left|\partial_x^s Du(t)\right|_2^2 \leqslant 2Z_0(t) + \left|\partial_x^{s-1} u_t(t)\right|_2^2. \tag{4.77}$$

By (4.62), noting that $\nu_2(1, s-1) = \nu_2(0, s+1)$, and recalling the definition of Z_0 in (4.75), we deduce that (with different C and ω)

$$\left|\partial_x^s Du(t)\right|_2^2 \leqslant \big(C + \omega(R)\Phi_s^4(t)\big)(1+t)^{-2\nu_2(0,s+1)}, \tag{4.78}$$

from which

$$\Phi_s(t) \leqslant C_s + \omega_s(R)\Phi_s^2(t), \tag{4.79}$$

for suitable $C_s \geqslant 1$ independent of R, and $\omega_s \in \mathcal{K}_0$. It is then immediate to show that, if R is sufficiently small, there is $C_2 > C_s$ such that, for all $t \geqslant 0$,

$$\Phi_s(t) \leqslant C_2; \tag{4.80}$$

thus, under the reservation that (4.72) holds, Φ_s is bounded. This implies (3.4) for $m = s$, with $C_{0,s} := C_2$.

2) We prove (3.6) for $m = 1$, and (3.5), (3.6) for $m = s - 1$. From (4.4) we derive that

$$\left|w_{tt}(t)\right|_2 \leqslant \left|g\big(u(t)\big)\right|_2 + C\int_0^t (1+t-\theta)^{-\nu_2(2,0)}\big(\left|\nabla g(u)\right|_2 + \left|g(u)\right|_2\big)\,\mathrm{d}\theta. \tag{4.81}$$

Since $s+1 \geqslant 4$ because $N \geqslant 3$, it follows that $\frac{s+1}{2} \geqslant 2 = \nu_2(2,0)$; hence, by (4.24) with $k = 1$, and by (4.80),

$$\left|g\big(u(\theta)\big)\right|_2 \leqslant \omega(1)C_2(1+\theta)^{-\nu_2(1,s-1)} \leqslant C(1+\theta)^{-\nu_2(2,0)}. \tag{4.82}$$

Next, we use interpolation between (4.29) with $k = 1$ and (4.82), to deduce, again via (4.80), that

$$\begin{aligned}\left|\nabla g(u)\right|_2 &\leqslant C\left|\partial_x^{s-1} g(u)\right|_2^{1/(s-1)}\left|g(u)\right|_2^{1-1/(s-1)}\\ &\leqslant C(1+\theta)^{-\nu_2(1,s-1)} \leqslant C(1+\theta)^{-\nu_2(2,0)}.\end{aligned} \tag{4.83}$$

Thus, we conclude from (4.81), via Lemma 2.1, that

$$\left|w_{tt}(t)\right|_2 \leqslant C(1+t)^{-\nu_2(2,0)}; \tag{4.84}$$

together with the analogous estimate for $|v_{tt}(t)|_2$, (4.84) implies (3.6) for $m = 1$.

Likewise, inserting (4.80) into (4.18) and (4.19), we deduce that

$$\left|\partial_x^{s-k}\partial_t^k u(t)\right|_2 \leqslant C(1+t)^{-\nu_2(k,s-k)}, \quad 1 \leqslant k \leqslant 3; \tag{4.85}$$

in particular, (3.5) and (3.6) hold for $m = s - 1$.

Step 4. We use an energy method, similar to the one we used in Step 3, to prove (3.5) and (3.6) for $m = s$. Let $k = 1, 2$. Recalling (4.63), we generalize (4.64) into

$$P_k^u(\cdot) := \left|\partial_x^{s-k}\partial_t^k u_t\right|_2^2 + \left\langle \partial_x^{s-k}\partial_t^k u, \partial_x^{s-k}\partial_t^k u_t\right\rangle + Q_{s-k}\big(\nabla\partial_t^k u\big); \tag{4.86}$$

we also set (compare to (4.17))

$$\Psi_k(t) := \max\Bigl\{1, \sup_{0\leqslant\theta\leqslant t}\bigl((1+\theta)^{\nu_2(k,s+1-k)}\bigl|\partial_x^{s-k}\partial_t^k Du(\theta)\bigr|_2\bigr)\Bigr\} \tag{4.87}$$

(thus, $\Psi_0 = \Phi_s$). We differentiate Eq. (1.1) first k times with respect to t, to obtain

$$\partial_t^k u_{tt} + \partial_t^k u_t - a_{ij}(Du)\partial_i\partial_j\partial_t^k u = \sum_{\ell=1}^{k}\binom{k}{\ell}\partial_t^\ell \tilde{a}_{ij}(Du)\partial_t^{k-\ell}\partial_i\partial_j u =: D_k(u); \tag{4.88}$$

then, we differentiate (4.88) α times with respect to x, $|\alpha| = s-k$, and multiply the resulting equations in L^2 by $2\partial_x^\alpha\partial_t^k u_t + \partial_x^\alpha\partial_t^k u$. Summing the resulting identities for $|\alpha| = s-k$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}P_k^u + P_k^u &= \sum_{|\alpha|=s-k}\bigl(R_{0,\alpha}^{(k)} + R_{1,\alpha}^{(k)} + R_{2,\alpha}^{(k)} + R_{G,1,\alpha}^{(k)} + R_{G,2,\alpha}^{(k)}\bigr)\\ &\quad + \sum_{|\alpha|=s-k}\bigl\langle\partial_x^\alpha D_k(u), \partial_x^\alpha\partial_t^k(2u_t+u)\bigr\rangle\\ &=: \Lambda_k + M_k,\end{aligned} \tag{4.89}$$

where the terms $R_{0,\alpha}^{(k)}, \dots, R_{G,2,\alpha}^{(k)}$, are defined in analogy to (4.66), (4.67), (4.68), (4.69) and (4.70); that is,

$$R_{0,\alpha}^{(k)} := \bigl\langle\partial_t\bigl(a_{ij}(Du)\bigr)\partial_i\partial_x^\alpha\partial_t^k u, \partial_j\partial_x^\alpha\partial_t^k u\bigr\rangle, \tag{4.90}$$

$$R_{1,\alpha}^{(k)} + R_{2,\alpha}^{(k)} := -\bigl\langle\partial_j\bigl(a_{ij}(Du)\bigr)\partial_i\partial_x^\alpha\partial_t^k u, \partial_x^\alpha\partial_t^k(2u_t+u)\bigr\rangle, \tag{4.91}$$

and

$$R_{G,1,\alpha}^{(k)} + R_{G,2,\alpha}^{(k)} := \sum_{0<\beta\leqslant\alpha}\binom{\alpha}{\beta}\bigl\langle\partial_x^\beta a_{ij}(Du)\partial_x^{\alpha-\beta}\partial_i\partial_j\partial_t^k u, \partial_x^\alpha\partial_t^k(2u_t+u)\bigr\rangle. \tag{4.92}$$

In Step 5 below, we shall prove that there are $\omega_\Lambda, \omega_M \in \mathcal{K}_0$ such that

$$\Lambda_k \leqslant \omega_\Lambda(R)\bigl|\partial_x^{s-k}\partial_t^k Du\bigr|_2^2 + C(1+t)^{-2\nu_2(k,s+1-k)} \tag{4.93}$$

for $k = 0, 1, 2$, and

$$M_k \leqslant \omega_M(R)\bigl|\partial_x^{s-k}\partial_t^k Du\bigr|_2^2 + C(1+t)^{-2\nu_2(k,s+1-k)} \tag{4.94}$$

for $k = 1, 2$. Assuming this for the moment, we deduce from (4.89) that, for $\omega = \omega_\Lambda + \omega_M$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigl(\mathrm{e}^t P_k^u\bigr) \leqslant \mathrm{e}^t\bigl(\omega(R)\bigl|\partial_x^{s-k}\partial_t^k Du\bigr|_2^2 + C(1+t)^{-2\nu_2(k,s+1-k)}\bigr); \tag{4.95}$$

from this, since Ψ_k is increasing,

$$\mathrm{e}^t P_k^u(t) \leqslant P_k^u(0) + \bigl(\omega(R)\Psi_k^2(t) + C\bigr)\int_0^t \mathrm{e}^\theta(1+\theta)^{-2\nu_2(k,s+1-k)}\,\mathrm{d}\theta. \tag{4.96}$$

Recalling that $|P_k^u(0)| \leqslant C\delta^2 \leqslant C$, by Lemma 2.2 we deduce from (4.96) that

$$P_k^u(t) \leqslant \bigl(C + \omega(R)\Psi_k^2(t)\bigr)(1+t)^{-2\nu_2(k,s+1-k)} =: Z_k(t). \tag{4.97}$$

Recalling (4.86), we further deduce from (4.97) that

$$\begin{aligned}\left|\partial_x^{s-k}\partial_t^k Du(t)\right|_2^2 &\leqslant Z_k(t)-\left\langle\partial_x^{s-k}\partial_t^k u(t),\partial_x^{s-k}\partial_t^k u_t(t)\right\rangle\\ &\leqslant Z_k(t)+\left\langle\partial_x^{s+1-k}\partial_t^k u(t),\partial_x^{s-1-k}\partial_t^k u_t(t)\right\rangle\\ &\leqslant Z_k(t)+\frac{1}{2}\left|\partial_x^{s+1-k}\partial_t^k u(t)\right|_2^2+\frac{1}{2}\left|\partial_x^{s-1-k}\partial_t^k u_t(t)\right|_2^2,\end{aligned} \tag{4.98}$$

from which

$$\begin{aligned}\left|\partial_x^{s-k}\partial_t^k Du(t)\right|_2^2 &\leqslant 2Z_k(t)+\left|\partial_x^{s-1-k}\partial_t^k u_t(t)\right|_2^2\\ &=2Z_k(t)+\left|\partial_x^{s-(k+1)}\partial_t^{k+1}u(t)\right|_2^2.\end{aligned} \tag{4.99}$$

By (4.85), with k replaced by $k+1$ (note that $k+1=2,3$),

$$\begin{aligned}\left|\partial_x^{s-(k+1)}\partial_t^{k+1}u(t)\right|_2 &\leqslant C(1+t)^{-\nu_2(k+1,s-(k+1))}\\ &=C(1+t)^{-\nu_2(k,s+1-k)}.\end{aligned} \tag{4.100}$$

Inserting (4.100) into (4.99), and recalling the definition of Z_k in (4.97), we obtain

$$\left|\partial_x^{s-k}\partial_t^k Du(t)\right|_2^2 \leqslant \left(C+\omega(R)\Psi_k^2(t)\right)(1+t)^{-2\nu_2(k,s+1-k)}, \tag{4.101}$$

from which

$$\Psi_k(t)\leqslant C+\omega(R)\Psi_k(t). \tag{4.102}$$

Taking R so small that $\omega(R)\leqslant\frac{1}{2}$, we conclude from (4.102) that

$$(1+t)^{\nu_2(k,s+1-k)}\left|\partial_x^{s+1-k}\partial_t^k u(t)\right|_2\leqslant C, \tag{4.103}$$

from which (3.5) and (3.6) follow, for $m=s$.

Step 5. We prove estimate (4.93); in fact, we prove more, that is, that, for $0\leqslant k\leqslant 2$, there is $\omega\in\mathcal{K}_0$ such that

$$\Lambda_k\leqslant\omega(R)\left|\partial_x^{s-k}\partial_t^k Du(t)\right|_2^2+\omega(R)\Phi_s^3(t)(1+t)^{-2\nu_2(k,s+1-k)}. \tag{4.104}$$

Taking $k=0$ in (4.104) yields (4.72), and, thus, (4.80); taking then $C=\omega(1)$ in the last term of (4.104) yields (4.93). At first, from (4.90) we immediately obtain

$$R_{0,\alpha}^{(k)}\leqslant\left|a'_{ij}(Du)\right|_\infty|Du_t|_\infty\left|\nabla\partial_x^\alpha\partial_t^k u\right|_2^2\leqslant\omega(R)\left|\partial_x^{s-k}\partial_t^k Du\right|_2^2. \tag{4.105}$$

Similarly, from (4.91),

$$R_{1,\alpha}^{(k)}\leqslant\omega(R)\left|\nabla\partial_x^\alpha\partial_t^k u\right|_2\left|\partial_x^\alpha\partial_t^{k+1}u\right|_2\leqslant\omega(R)\left|\partial_x^{s-k}\partial_t^k Du\right|_2^2, \tag{4.106}$$

and, for $\theta=\frac{N}{2(s-1)}\in\left]\frac{1}{s-1},1\right[$,

$$\begin{aligned}R_{2,\alpha}^{(k)} &\leqslant\left|a'_{ij}(Du)\right|_\infty|D\partial_j u|_N\left|\nabla\partial_x^\alpha\partial_t^k u\right|_2\left|\partial_x^\alpha\partial_t^k u\right|_{\frac{2N}{N-2}}\\ &\leqslant\kappa(R)\left|\partial_x^{s-1}Du\right|_2^\theta|Du|_2^{1-\theta}\left|\partial_x^{s-k}\partial_t^k\nabla u\right|_2^2\\ &\leqslant\omega(R)\left|\partial_x^{s-k}\partial_t^k Du\right|_2^2.\end{aligned} \tag{4.107}$$

Next, for suitable $p,q\in[2,+\infty]$, such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$, and noting that, in the sum of (4.92), $b:=|\beta|\geqslant 1$, by Proposition 2.1 again,

$$
\begin{aligned}
R_{G,1,\alpha}^{(k)} &\leqslant C \sum_{0<\beta\leqslant\alpha} \left|\partial_x^\beta a_{ij}(Du)\right|_p \left|\partial_x^{\alpha-\beta}\partial_i\partial_j\partial_t^k u\right|_q \left|\partial_x^\alpha\partial_t^k u_t\right|_2 \\
&\leqslant \kappa(R) \sum_{b=1}^{s-k} \left|\partial_x^b Du\right|_p \left|\partial_x^{s+2-k-b}\partial_t^k u\right|_q \left|\partial_x^{s-k}\partial_t^k u_t\right|_2 \\
&\leqslant C\left|\partial_x^s Du\right|_2^\lambda |Du|_2^{1-\lambda} \left|\partial_x^{s-k}\partial_t^k \nabla u\right|_2^\mu \left|\partial_t^k u\right|_2^{1-\mu} \left|\partial_x^{s-k}\partial_t^k u_t\right|_2,
\end{aligned}
\tag{4.108}
$$

with

$$
\frac{1}{p} = \frac{b}{N} + \frac{1}{2} - \lambda\frac{s}{N}, \qquad \frac{1}{q} = \frac{s+2-k-b}{N} + \frac{1}{2} - \mu\frac{s+1-k}{N}. \tag{4.109}
$$

We now show that

$$
\left|\partial_t^k u(t)\right|_2 \leqslant C(1+t)^{-\frac{N}{4}}. \tag{4.110}
$$

Indeed, for $k=0$ and $k=1$ (4.110) follows, respectively, from (3.7) and (4.6); for $k=2$ (note that we cannot yet use (3.6), because the validity of this estimate is subject to the validity of (4.72), which is part of what we are proving in this step), using Eq. (1.1), and (4.6) again, we estimate

$$
\begin{aligned}
|u_{tt}|_2 &\leqslant |u_t|_2 + \left|a_{ij}(Du)\right|_\infty |\partial_i\partial_j u|_2 \\
&\leqslant |u_t|_2 + \kappa(R)\left|\partial_x^s \nabla u\right|_2^{1/s} |\nabla u|_2^{1-1/s} \\
&\leqslant C(1+t)^{-(\frac{N}{4}+\frac{1}{2})} + \omega(R)(1+t)^{-(1-\frac{1}{s})(\frac{N}{4}+\frac{1}{2})};
\end{aligned}
\tag{4.111}
$$

and since $(1-\frac{1}{s})(\frac{N}{4}+\frac{1}{2}) \geqslant \frac{N}{4}$ (because $s > \frac{N}{2}+1$), (4.111) yields (4.110) for $k=2$. Thus, we obtain from (4.108) that, for $\varepsilon \in\,]0,\lambda[$ to be determined,

$$
\begin{aligned}
R_{G,1,\alpha}^{(k)} &\leqslant \omega(R)\left|\partial_x^s Du\right|_2^{\lambda-\varepsilon} |Du|_2^{1-\lambda} \left|\partial_t^k u\right|_2^{1-\mu} \left|\partial_x^{s-k}\partial_t^k Du\right|_2^{1+\mu} \\
&\leqslant \omega(R)\left|\partial_x^{s-k}\partial_t^k Du\right|_2^2 + \omega(R)\Phi_s^{2(\lambda-\varepsilon)/(1-\mu)}(t)(1+t)^{-\nu},
\end{aligned}
\tag{4.112}
$$

where

$$
\nu = (s+1)\frac{\lambda-\varepsilon}{1-\mu} + \frac{1-\lambda}{1-\mu}\left(\frac{N}{2}+1\right) + \frac{N}{2}. \tag{4.113}
$$

Using (4.109), it is not difficult to verify that, if we define ε by

$$
\frac{\lambda-\varepsilon}{1-\mu} = 1 + \frac{1}{2(s+1)}, \tag{4.114}
$$

then $\varepsilon \in\,]0,\lambda[$. With this choice of ε, and recalling that $N \geqslant 3$, (4.113) yields that

$$
\nu \geqslant (s+1)\left(1+\frac{1}{2(s+1)}\right) + \frac{N}{2} \geqslant s+3 \geqslant s+1+k = \nu_2(k,s+1-k). \tag{4.115}
$$

Since also $\frac{2(\lambda-\varepsilon)}{1-\mu} = 2 + \frac{1}{s+1} < 3$, we finally conclude from (4.112) that

$$
R_{G,1,\alpha}^{(k)} \leqslant \omega(R)\left|\partial_x^{s-k}\partial_t^k Du\right|_2^2 + \omega(R)\Phi_s^3(t)(1+t)^{-\nu_2(k,s+1-k)}, \tag{4.116}
$$

as desired in (4.104). We now wish to estimate, as in (4.107) and (4.108),

$$R^{(k)}_{G,2,\alpha} \leqslant \kappa(R) \sum_{b=1}^{s-k} \left|\partial_x^b Du\right|_p \left|\partial_x^{s+2-k-b} \partial_t^k u\right|_q \left|\partial_x^{s-k} \partial_t^k u\right|_{\frac{2N}{N-2}}, \tag{4.117}$$

for $p, q \in]2, +\infty[$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{N}$ (considering $\partial_x^b Du \in H^{s-b} \hookrightarrow L^p$ and $\partial_x^{s+2-k-b} \partial_t^k u \in H^{b-1} \hookrightarrow L^q$). From (4.117), we proceed as in (4.112), that is

$$\begin{aligned} R^{(k)}_{G,2,\alpha} &\leqslant \omega(R) \left|\partial_x^s Du\right|_2^{\lambda-\varepsilon} |Du|_2^{1-\lambda} \left|\partial_t^k u\right|_2^{1-\mu} \left|\partial_x^{s-k} \partial_t^k Du\right|_2^{1+\mu} \\ &\leqslant \omega(R) \left|\partial_x^{s-k} \partial_t^k Du\right|_2^2 + \omega(R) \Phi_s^{2(\lambda-\varepsilon)/(1-\mu)}(t)(1+t)^{-\nu}, \end{aligned} \tag{4.118}$$

with the same λ and μ of (4.109), and the same ν and ε of (4.113) and (4.114). The only difference with the previous case is that, now,

$$\lambda + \mu = 1 + \frac{N}{2s} - \frac{k-1}{s} + \frac{k-1}{s}\mu. \tag{4.119}$$

Thus, from (4.118) we deduce that

$$R^{(k)}_{G,2,\alpha} \leqslant \omega(R) \left|\partial_x^{s-k} \partial_t^k Du\right|_2^2 + \omega(R) \Phi_s^3(t)(1+t)^{-\nu_2(k,s+1-k)}, \tag{4.120}$$

again as desired in (4.104). In conclusion, inserting (4.105), (4.106), (4.107), (4.116) and (4.120) into the definition of Λ_k in (4.89), we see that (4.104) follows. In particular, Λ_0 satisfies (4.72); thus, Step 3 is now complete, and (4.80) can be assumed to hold. As we have seen in the second part of Step 3, this implies the validity of (4.85), and of (3.6) for $m = 1$; as a consequence, (3.4) holds for $m \leqslant s$; in addition, (3.5) and (3.6) hold at least for $m \leqslant s - 1$.

Omitting the proof of (4.94), which follows with a procedure very similar to that of Step 5, this ends the proof of Theorem 3.1.

5. Proof of Theorem 3.2

The proof of Theorem 3.2 follows essentially the same lines of that of Theorem 3.1. Here too, if δ is sufficiently small, Theorem 2.2 implies that the Cauchy problem (1.2) + (1.4) does have a global solution $u \in \mathcal{P}_s$; in addition, the quantity $R := \sup_{t \geqslant 0} N_p(t)$ can be made as small as desired, by taking δ conveniently small; in particular, we can assume that $R \leqslant 1$. We also note that (3.9) follows from (3.8), using Eq. (1.2) itself, and keeping in mind the identity $\nu_2(1, m) = \nu_2(0, m + 2)$; more precisely, if (3.8) holds for $1 \leqslant m \leqslant s$, then (3.9) holds for $0 \leqslant m \leqslant s - 1$.

To prove (3.8), as in (4.1) we rewrite (1.2) in the linearized form

$$u_t - a_{ij}(0)\partial_i \partial_j u = h(u) := \tilde{a}_{ij}(\nabla u)\partial_i \partial_j u, \tag{5.1}$$

and note that, by the linear estimates (2.13) and (2.14), it is sufficient to estimate the difference $w = u - v$, where $v(t) := u_{u_0}$ and

$$w(t) := \int_0^t u_{h(u(\theta))}(t-\theta)\, \mathrm{d}\theta. \tag{5.2}$$

In analogy with (4.17), for $0 \leqslant m \leqslant s$ and $t > 0$ we define

$$\Phi_m(t) := \max\left\{1, \sup_{0 \leqslant \theta \leqslant t} \left((1+\theta)^{\nu_2(0,m+1)} \left\|\partial_x^m \nabla u(\theta)\right\|\right)\right\}, \tag{5.3}$$

and seek to establish time-independent bounds on $\Phi_0(t)$ and $\Phi_s(t)$ (the other cases following by interpolation).

1. We set $\Phi(t) := \Phi_0(t) + \Phi_{s-1}(t)$, and $\ell := 2 - \frac{1}{s-1}$. By the linear estimates (2.14), we deduce from (5.2) that

$$\big|\nabla w(t)\big|_2 \leqslant C \int_0^t (1+t-\theta)^{-\nu_1(0,1)} \big(\big|h\big(u(\theta)\big)\big|_1 + \big|\nabla h\big(u(\theta)\big)\big|_2\big)\,\mathrm{d}\theta. \tag{5.4}$$

By Propositions 2.2 and 2.1, as in (4.9),

$$\begin{aligned}
\big|h(u)\big|_1 &\leqslant \big|\tilde{a}_{ij}(\nabla u)\big|_2 |\partial_i \partial_j u|_2 \leqslant \kappa(R)|\nabla u|_2 \big|\partial_x^2 u\big|_2 \\
&\leqslant C|\nabla u|_2 \big|\partial_x^s u\big|_2^{\frac{1}{s-1}} |\nabla u|_2^{1-\frac{1}{s-1}} \leqslant \omega(R)|\nabla u|_2^\ell \\
&\leqslant \omega(R)\Phi_0^\ell(\theta)(1+\theta)^{-\ell/2} \leqslant \omega(R)\Phi_0^\ell(\theta)(1+\theta)^{-1/2},
\end{aligned} \tag{5.5}$$

with $\omega \in \mathcal{K}_0$. Likewise, with some abuse of notation,

$$\big|\nabla\big(h(u)\big)\big|_2 \leqslant \big|\tilde{a}'_{ij}(\nabla u)\partial_x \nabla u \partial_i \partial_j u\big|_2 + \big|\tilde{a}_{ij}(\nabla u)\partial_i \partial_j \nabla u\big|_2 =: H_1 + H_2. \tag{5.6}$$

Setting $\eta := \frac{N+4}{4(s-1)} \in\,]0,1[$, we estimate

$$\begin{aligned}
H_1 &\leqslant \kappa(R)\big|\partial_x^2 u\big|_4^2 \leqslant C\big|\partial_x^s u\big|_2^{2\eta} |\nabla u|_2^{2(1-\eta)} \\
&\leqslant C R^\eta \Phi_{s-1}^\eta(\theta)(1+\theta)^{-\eta s/2} \Phi_0^{2(1-\eta)}(\theta)(1+\theta)^{-(1-\eta)} \\
&\leqslant \omega(R)\Phi_{s-1}^\eta(\theta)\Phi_0^{2(1-\eta)}(\theta)(1+\theta)^{-\rho},
\end{aligned} \tag{5.7}$$

with $\rho := \eta\frac{s}{2} + 1 - \eta$. Similarly, for p and $q \geqslant 1$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$,

$$H_2 \leqslant \kappa(R)|\nabla u|_p \big|\partial_x^3 u\big|_q \leqslant C\big|\partial_x^s u\big|_2^\lambda |\nabla u|_2^{2-\lambda}, \tag{5.8}$$

with $\lambda = \frac{N+4}{2(s-1)} = 2\eta$. Thus, H_2 satisfies the same estimate (5.7) as H_1; hence, recalling (5.6),

$$\big|\nabla h\big(u(\theta)\big)\big|_2 \leqslant \omega(R)\Phi_{s-1}^\eta(\theta)\Phi_0^{2(1-\eta)}(\theta)(1+\theta)^{-\rho}. \tag{5.9}$$

Inserting (5.5) and (5.9) into (5.4), by Lemma 2.1 we obtain that

$$\begin{aligned}
\big|\nabla w(t)\big|_2 &\leqslant \omega(R)\int_0^t (1+t-\theta)^{-\nu_1(0,1)} \Phi_0^\ell(\theta)(1+\theta)^{-1/2}\,\mathrm{d}\theta \\
&\qquad + \omega(R)\int_0^t (1+t-\theta)^{-\nu_1(0,1)} \Phi_{s-1}^\eta(\theta)\Phi_0^{2(1-\eta)}(\theta)(1+\theta)^{-\rho}\,\mathrm{d}\theta \\
&\leqslant \omega(R)\Phi_0^\ell(t)(1+t)^{-1/2} + \omega(R)\Phi_{s-1}^\eta(t)\Phi_0^{2(1-\eta)}(t)(1+t)^{-\mu},
\end{aligned} \tag{5.10}$$

where $\mu := \min\{\nu_1(0,1), \rho\} \geqslant \frac{1}{2}$. Thus, since v satisfies the linear estimate

$$\big|\nabla v(t)\big|_2 \leqslant C_L(1+t)^{-1/2}, \tag{5.11}$$

for suitable $C_L > 0$ depending on u_0, we conclude that

$$\Phi_0(t) \leqslant C_L + \omega(R)\Phi_0^{\ell}(t) + \omega(R)\Phi_{s-1}^{\eta}(t)\Phi_0^{2(1-\eta)}(t). \tag{5.12}$$

Noting that $2(1-\eta) < 2 - \frac{1}{s-1} = \ell$, we further obtain from (5.12) that

$$\Phi_0(t) \leqslant C_L + \omega(R)\Phi_0^{\ell}(t) + \omega(R)\Phi_{s-1}^{r}(t) =: \Psi(t), \tag{5.13}$$

with $r = \frac{\ell\eta}{\ell - 2(1-\eta)} \leqslant \ell$.

2. We now use an energy method to obtain an intermediate estimate on $\Phi_{s-1}(t)$. We differentiate Eq. (1.2) α times with respect to x, $|\alpha| = s$, and multiply the resulting equation in L^2 by $2(1+t)^n\partial_x^{\alpha}u$, with $n \geqslant s$. Summing in α, and recalling (4.63), we obtain that, for all $t \geqslant 0$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big((1+t)^n\big|\partial_x^s u\big|_2^2\big) + 2(1+t)^n Q_s(\nabla u) \\
&\quad = n(1+t)^{n-1}\big|\partial_x^s u\big|_2^2 + 2(1+t)^n \sum_{|\alpha|=s}(R_{2,\alpha} + R_{G,2,\alpha}),
\end{aligned} \tag{5.14}$$

where $R_{2,\alpha}$ and $R_{G,2,\alpha}$, are defined in (4.68) and (4.70), but with Du replaced by ∇u in the coefficients a_{ij}. For $\theta \in]0,1[$ to be chosen later, we estimate

$$\begin{aligned}
R_{2,\alpha} &\leqslant C\big|a_{ij}'(\nabla u)\big|_\infty|\partial_j\nabla u|_\infty\big|\nabla\partial_x^s u\big|_2\big|\partial_x^s u\big|_2 \\
&\leqslant \omega(R)\big|\nabla\partial_x^s u\big|_2\big(\big|\partial_x^2 u\big|_\infty\big|\partial_x^s u\big|_2\big)^{\theta}.
\end{aligned} \tag{5.15}$$

By Proposition 2.1, we can proceed with

$$R_{2,\alpha} \leqslant \omega(R)\big|\partial_x^s\nabla u\big|_2^{1+\theta(\sigma_1+\sigma_2)}|\nabla u|_2^{\theta(2-\sigma_1-\sigma_2)}, \tag{5.16}$$

with $\sigma_1 = \frac{N+2}{2s}$ and $\sigma_2 = \frac{s-1}{s}$. We now choose θ so that $\theta(\sigma_1+\sigma_2) = 1$; since $1 < \sigma_1 + \sigma_2 < 2$, it follows that $\frac{1}{2} < \theta < 1$, and $\theta(2-\sigma_1-\sigma_2) = 2\theta - 1 > 0$. Hence, we obtain from (5.16) that

$$R_{2,\alpha} \leqslant \omega(R)\big|\nabla\partial_x^s u\big|_2^2. \tag{5.17}$$

Likewise, for suitable $p, q, r \geqslant 1$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, by Propositions 2.2 (keep in mind that $\beta > 0$) and 2.1 again,

$$\begin{aligned}
R_{G,2,\alpha} &\leqslant C\sum_{0<\beta\leqslant\alpha}\big|\partial_x^{|\beta|}a_{ij}(\nabla u)\big|_p\big|\partial_x^{s-|\beta|+1}\nabla u\big|_q\big|\partial_x^s u\big|_r \\
&\leqslant \kappa(R)\sum_{0<\beta\leqslant\alpha}\underbrace{\big|\partial_x^{|\beta|}\nabla u\big|_p\big|\partial_x^{s-|\beta|+1}\nabla u\big|_q\big|\partial_x^s u\big|_r}_{=:S_\beta} \\
&\leqslant \kappa(R)\big|\partial_x^s\nabla u\big|_2^{\lambda}|\nabla u|_2^{3-\lambda},
\end{aligned} \tag{5.18}$$

with $\lambda := 2 + \frac{N}{2s} \in]2,3[$. Thus,

$$\begin{aligned}
R_{G,2,\alpha} &\leqslant \kappa(R)\sum_{0<\beta\leqslant\alpha}S_\beta^{1-2/\lambda}S_\beta^{2/\lambda} \\
&\leqslant \omega(R)\big|\partial_x^s\nabla u\big|_2^2|\nabla u|_2^{2(3-\lambda)/\lambda} \leqslant \omega(R)\big|\nabla\partial_x^s u\big|_2^2.
\end{aligned} \tag{5.19}$$

Inserting (5.17) and (5.19) into (5.14), and recalling that $\alpha_0 \geqslant 1$, we deduce that, for $t \geqslant 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big((1+t)^n\big|\partial_x^s u\big|_2^2\big) + 2(1+t)^n\big|\nabla\partial_x^s u\big|_2^2 \leqslant n(1+t)^{n-1}\big|\partial_x^s u\big|_2^2 + \omega(R)(1+t)^n\big|\nabla\partial_x^s u\big|_2^2; \tag{5.20}$$

thus, if R is so small that $\omega(R) \leqslant 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big((1+t)^n\big|\partial_x^s u\big|_2^2\big) + (1+t)^n\big|\nabla\partial_x^s u\big|_2^2 \leqslant n(1+t)^{n-1}\big|\partial_x^s u\big|_2^2. \tag{5.21}$$

Again by interpolation, with $\sigma_2 = 1 - \frac{1}{s}$, and recalling (5.13),

$$\begin{aligned} n(1+t)^{n-1}\big|\partial_x^s u\big|_2^2 &\leqslant C(1+t)^{n-1}\big|\nabla\partial_x^s u\big|_2^{2\sigma_2}|\nabla u|_2^{2(1-\sigma_2)} \\ &\leqslant C(1+t)^{n\sigma_2}\big|\nabla\partial_x^s u(t)\big|_2^{2\sigma_2}(1+t)^{(n-1)(1-\sigma_2)-1}\Psi^{2(1-\sigma_2)}(t) \\ &\leqslant (1+t)^n\big|\nabla\partial_x^s u(t)\big|_2^2 + C(1+t)^{n-1-1/(1-\sigma_2)}\Psi^2(t). \end{aligned} \tag{5.22}$$

Inserting (5.22) into (5.21) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\big((1+t)^n\big|\partial_x^s u\big|_2^2\big) \leqslant C(1+t)^{n-1-s}\Psi^2(t); \tag{5.23}$$

thus, fixing $n > s$ and integrating, we obtain that, since Ψ is increasing,

$$(1+t)^n\big|\partial_x^s u(t)\big|_2^2 \leqslant \big|\partial_x^s u(0)\big|_2^2 + C\Psi^2(t)(1+t)^{n-s}, \tag{5.24}$$

from which

$$(1+t)^s\big|\partial_x^s u(t)\big|_2^2 \leqslant C_1 + \Psi^2(t), \tag{5.25}$$

and, finally, recalling (5.13),

$$\Phi_{s-1}(t) \leqslant C_2 + \Psi(t) \leqslant C_3 + \omega(R)\Phi_0^\ell(t) + \omega(R)\Phi_{s-1}^r(t). \tag{5.26}$$

Adding (5.13) to (5.26), and recalling that $\ell \geqslant r$ and $\Phi_{s-1} \geqslant 1$, we deduce that

$$\Phi(t) \leqslant C_4 + \omega(R)\big(\Phi_0^\ell(t) + \Phi_{s-1}^\ell(t)\big) \leqslant C_4 + \omega(R)\Phi^\ell(t), \tag{5.27}$$

from which it follows that, if R is sufficiently small, there is $C_0 > 0$ such that, for all $t \geqslant 0$,

$$\Phi(t) \leqslant C_0. \tag{5.28}$$

Thus, Φ is bounded; therefore, so are Φ_0 and Φ_{s-1}; that is, (3.8) holds for $m = 0$ and $m = s-1$. As we have remarked above, this also implies that (3.9) holds for $0 \leqslant m \leqslant s-2$.

3. We now prove (3.8) for $m = s$, under the additional assumption that $u_0 \in L^1$. To this end, we first note that, in this case, the decay estimate (3.8) for $m = 0$ can be improved into

$$\big|\nabla u(t)\big|_2 \leqslant C_1(1+t)^{-\ell/2}. \tag{5.29}$$

Indeed, if $u_0 \in L^1$, then v satisfies the linear estimate

$$\big|\nabla v(t)\big|_2 \leqslant C_L(1+t)^{-\nu_1(0,1)}; \tag{5.30}$$

on the other hand, since Φ_0 and Φ_{s-1} are bounded, we deduce from the third line of (5.5) and from (5.9) that, since $\rho \geqslant \frac{\ell}{2}$,

$$\left|h(u)\right|_1 + \left|\nabla h(u)\right|_2 \leqslant C(1+t)^{-\min\{\rho,\ell/2\}} = C(1+t)^{-\ell/2}. \tag{5.31}$$

Since also $\nu_1(0,1) \geqslant \frac{\ell}{2}$, we deduce from (5.4) that

$$\left|\nabla w(t)\right|_2 \leqslant C(1+t)^{-\min\{\nu_1(0,1),\ell/2\}} = C(1+t)^{-\ell/2}. \tag{5.32}$$

Together with (5.30), (5.32) yields (5.29). Inserting this into (5.22), we can modify the latter into

$$\begin{aligned} n(1+t)^{n-1}\left|\partial_x^s u\right|_2^2 &\leqslant C(1+t)^{n\sigma_2}\left|\nabla\partial_x^s u\right|_2^{2\sigma_2}(1+t)^{(n-\ell)(1-\sigma_2)-1} \\ &\leqslant \frac{1}{2}(1+t)^n\left|\nabla\partial_x^s u\right|_2^2 + C(1+t)^{n-\ell-s}. \end{aligned} \tag{5.33}$$

In turn, (5.33) allows us to modify (5.23) into

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((1+t)^n\left|\partial_x^s u\right|_2^2\right) + \frac{1}{2}(1+t)^n\left|\nabla\partial_x^s u\right|_2^2 \leqslant C(1+t)^{n-\ell-s}, \tag{5.34}$$

from which, choosing $n = s$ and integrating, we obtain that

$$\int_0^t (1+\theta)^s\left|\nabla\partial_x^s u\right|_2^2\,\mathrm{d}\theta \leqslant 2\left|\partial_x^s u(0)\right|_2^2 + \frac{C}{1-\ell}\left((1+t)^{1-\ell} - 1\right). \tag{5.35}$$

We now note that $1-\ell<0$, because the condition $\ell = 2 - \frac{1}{s-1} > 1$ is implied by the fact that $s \geqslant 3$. Hence, we finally conclude form (5.35) that

$$\int_0^t (1+\theta)^s\left|\nabla\partial_x^s u\right|_2^2\,\mathrm{d}\theta \leqslant C_1. \tag{5.36}$$

Our next step is to multiply the differentiated equations in L^2 by $2(1+t)^{s+1}\partial_x^\alpha u_t$, $|\alpha| = s$. We obtain

$$\begin{aligned} &2(1+t)^{s+1}\left|\partial_x^s u_t\right|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left((1+t)^{s+1}Q_s(\nabla u)\right) \\ &\quad = (s+1)(1+t)^s Q_s(\nabla u) + (1+t)^{s+1}\sum_{|\alpha|=s}(R_{0,\alpha} + R_{1,\alpha} + R_{G,1,\alpha}). \end{aligned} \tag{5.37}$$

We estimate the last terms of (5.37) as follows. At first, from Eq. (1.2) itself,

$$|u_t|_2 \leqslant \left|a_{ij}(\nabla u)\right|_\infty |\partial_i\partial_j u|_2 \leqslant \kappa(R)R = \omega(R); \tag{5.38}$$

next, for $\gamma_1 := \frac{N+2}{2s}$, $\gamma_2 := \frac{4}{2-\gamma_1}$ and $\gamma_3 := \frac{2(s+1)}{2-\gamma_1}$, and recalling (5.38),

$$\begin{aligned} R_{0,\alpha} &\leqslant h\left(|\nabla u|_\infty\right)|\nabla u_t|_\infty\left|\nabla\partial_x^s u\right|_2^2 \\ &\leqslant \kappa(R)\left|\partial_x^s u_t\right|_2^{\gamma_1}|u_t|_2^{1-\gamma_1}\left|\nabla\partial_x^s u\right|_2^2 \\ &\leqslant \omega(R)\left|\nabla\partial_x^s u\right|_2^{\frac{4}{2-\gamma_1}} + \frac{1}{3}\left|\partial_x^s u_t\right|_2^2 \\ &\leqslant \omega(R)\Phi_s^{\gamma_2}(t)(1+t)^{-\frac{2(s+1)}{2-\gamma_1}} + \frac{1}{3}\left|\partial_x^s u_t\right|_2^2 \\ &\leqslant \omega(R)\left(1+\Phi_s^{2(1+\gamma_1)}(t)\right)(1+t)^{-\gamma_3} + \frac{1}{3}\left|\partial_x^s u_t\right|_2^2, \end{aligned} \tag{5.39}$$

having noted that $\gamma_2 \leqslant 2(1+\gamma_1)$. Next, in a similar way, since $(1+\gamma_1)(s+1) \geqslant \gamma_3$,

$$\begin{aligned}
R_{1,\alpha} + R_{G,1,\alpha} &\leqslant h\big(|\nabla u|_\infty\big)|\nabla \partial_x u|_\infty \big|\nabla \partial_x^s u\big|_2 \big|\partial_x^s u_t\big|_2 \\
&\leqslant \kappa(R)|\nabla u|_2^{1-\gamma_1} \big|\nabla \partial_x^s u\big|_2^{1+\gamma_1} \big|\partial_x^s u_t\big|_2 \\
&\leqslant \omega(R)\big|\nabla \partial_x^s u\big|_2^{2(1+\gamma_1)} + \frac{2}{3}\big|\partial_x^s u_t\big|_2^2 \\
&\leqslant \omega(R)\Phi_s^{2(1+\gamma_1)}(t)(1+t)^{-(1+\gamma_1)(s+1)} + \frac{2}{3}\big|\partial_x^s u_t\big|_2^2 \\
&\leqslant \omega(R)\Phi_s^{2(1+\gamma_1)}(t)(1+t)^{-\gamma_3} + \frac{2}{3}\big|\partial_x^s u_t\big|_2^2.
\end{aligned} \tag{5.40}$$

Inserting (5.39) and (5.40) into (5.37), we obtain

$$\begin{aligned}
&(1+t)^{s+1}\big|\partial_x^s u_t\big|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t}\big((1+t)^{s+1} Q_s(\nabla u)\big) \\
&\quad \leqslant (s+1)(1+t)^s Q_s(\nabla u) + \omega(R)\big(1+\Phi_s^{2(1+\gamma_1)}(t)\big)(1+t)^{s+1-\gamma_3},
\end{aligned} \tag{5.41}$$

from which, integrating,

$$\begin{aligned}
&\int_0^t (1+\theta)^{s+1}\big|\partial_x^s u_t\big|_2^s \,\mathrm{d}\theta + (1+t)^{s+1} Q_s\big(\nabla u(t)\big) \\
&\quad \leqslant Q_s\big(\nabla u(0)\big) + (s+1)\int_0^t (1+\theta)^s Q_s(\nabla u)\,\mathrm{d}\theta \\
&\qquad + \omega(R)\big(1+\Phi_s^{2(1+\gamma_1)}(t)\big)\int_0^t (1+\theta)^{s+1-\gamma_3}\,\mathrm{d}\theta.
\end{aligned} \tag{5.42}$$

Recalling (5.36), and noting that $\gamma_3 - (s+1) > 1$, we deduce then from (5.42) that, in particular,

$$(1+t)^{s+1}\big|\nabla \partial_x^s u(t)\big|_2^s \leqslant C_2 + \omega(R)\Phi_s^{2(1+\gamma_1)}(t), \tag{5.43}$$

with C_2 depending also on C_1. In turn, we deduce from (5.43) that

$$\Phi_s(t) \leqslant C + \omega(R)\Phi_s^{1+\gamma_1}(t), \tag{5.44}$$

an inequality qualitatively similar to (5.27). Thus, we can deduce, as in (5.28), that, if R is sufficiently small, the map $t \mapsto \Phi_s(t)$ is bounded, and (3.8) for $m = s$ follows; consequently, (3.9) for $m = s-1$ also follows.

4. Omitting the proof of (3.10), which follows along similar lines, this concludes the proof of Theorem 3.2.

Remarks. In contrast to the hyperbolic case, we are not able to establish estimates for the higher order derivatives of u_t, similar to those of Theorem 3.1. This is due to the qualitative difference between the linear parabolic estimates of Theorem 2.5 and the hyperbolic ones of Theorem 2.3, which does not allow us to estimate the higher order derivatives of the difference $w_t = u_t - v_t$ via Duhamel's formula. Indeed, if we try to exploit the minimal regularity in the initial value, reflected in the presence of only the term $|w_0|_q$ in (2.13), we encounter the problem of the integrability of the function $t \mapsto t^{-\nu_q(k,m)}$ at $t = 0$, which essentially requires q to be close to 1, $k = 0$, and $m \leqslant 1$. On the other hand, if we try to remedy this by using (2.14), we need the corresponding higher regularity of the initial value. This is also reflected in the fact that the solution of the hyperbolic equation satisfies $u_t \in C_{\mathrm{b}}([0, +\infty[; H^s)$, while the solution of the parabolic equation satisfies $u_t \in C_{\mathrm{b}}([0, +\infty[; H^{s-1})$ only. Thus, a full extension of the results of Theorem 3.1 to the parabolic case is not to be expected. On the other hand, we point out that Theorem 3.2 holds for all $s > \frac{N}{2} + 1$, and not just for $\frac{N}{2} + 1 < s \leqslant N$, as we had to assume in Theorem 3.1.

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